DYADIC HARMONIC ANALYSIS
BEYOND DOUBLING MEASURES

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ABSTRACT. We characterize the Borel measures μ on $\mathbb{R}$ for which the associated dyadic Hilbert transform, or its adjoint, is of weak-type $(1, 1)$ and/or strong-type (p, p) with respect to μ. Surprisingly, the class of such measures is strictly bigger than the traditional class of dyadically doubling measures and strictly smaller than the whole Borel class. In higher dimensions, we provide a complete characterization of the weak-type $(1, 1)$ for arbitrary Haar shift operators, cancellative or not, written in terms of two generalized Haar systems and these include the dyadic paraproducts. Our main tool is a new Calderón-Zygmund decomposition valid for arbitrary Borel measures which is of independent interest.

1. INTRODUCTION

Dyadic techniques are nowadays fundamental in harmonic analysis. Their origin goes back to Hardy, Littlewood, Paley and Walsh among others. In the context of martingale inequalities, the dyadic maximal and square functions arise as particular cases of Doob's maximal function and Burkholder's square function for martingales associated to a dyadic filtration. Similarly, singular integral operators have been traditionally modeled by martingale transforms or martingale paraproducts. These last operators can be written in terms of martingale differences and conditional expectations, so that the full strength of probability methods applies in the analysis of their boundedness properties. In the Euclidean setting, dyadic martingale differences decompose as a sum of Haar functions and therefore we can obtain expansions using the classical Haar system.

In the last years dyadic operators have attracted a lot of attention related to the so-called A_2-conjecture. This seeks to establish that some operators obey an $L^2(w)$ estimate for every $w \in A_2$ with a constant that grows linearly in the A_2-characteristic of w. For the maximal function this was proved by Buckley [1]. In [24], Wittwer proved the A_2-conjecture for Haar multipliers in one dimension. The Beurling-Ahlfors transform, the Hilbert transform and the Riesz transforms were then considered by Petermichl and Volberg in [21], [19], [20] (see also [8]) and the A_2-conjecture for them was shown via the representation of these operators as averages of Haar multipliers and certain

dyadic operators called Haar shifts. The final solution to the A_2-conjecture for general Calderón-Zygmund operators was obtained by Hytönen in his celebrated paper [13]. Again, a key ingredient in the proof is that Calderón-Zygmund operators can be written as averages of dyadic operators including Haar shift operators, dyadic paraproducts and their adjoints.

The *dyadic Hilbert transform* is given by

$$H_{\mathscr{D}}f(x) = \sum_{I \in \mathscr{D}} \langle f, h_I \rangle (h_{I_-}(x) - h_{I_+}(x)).$$

Here $\mathscr{D}$ denotes some dyadic grid in $\mathbb{R}$ and h_I is the Haar function associated with $I \in \mathscr{D}$: $h_I = |I|^{-1/2}(1_{I_-} - 1_{I_+})$ where I_- and I_+ are the left and right dyadic children of I. The importance of this operator comes from the fact that the classical Hilbert transform can be obtained via averaging $H_{\mathscr{D}}$ over dyadic grids, this was shown by Petermichl [18]. That $H_{\mathscr{D}}$ is bounded on $L^2(\mathbb{R})$ follows easily from the orthogonality of the Haar system. Using the standard Calderón-Zygmund decomposition one can easily obtain (see for instance [5]) that $H_{\mathscr{D}}$ is of weak-type $(1,1)$ and therefore bounded on $L^p(\mathbb{R})$ for $1 < p < 2$. The bounds for $p > 2$ can be derived by duality and interpolation from the weak-type $(1,1)$ of the adjoint operator.

Let us consider a Borel measure μ in $\mathbb{R}$. One can define a Haar system in a similar manner which is now orthonormal in $L^2(\mu)$. Hence, we may consider a dyadic Hilbert transform which we momentarily denote by $H_{\mathscr{D}}^{\mu}$ and ask about its boundedness properties. The boundedness on $L^2(\mu)$ is again automatic by orthogonality. The standard Calderón-Zygmund theory can be easily extended to settings where the underlying measure is doubling. In the present situation, since the operator is dyadic, one could even relax that condition and assume that μ is dyadically doubling. In such a case, we can almost copy verbatim the standard proof and conclude the weak-type $(1,1)$ (with respect to μ) and therefore obtain the same bounds as before. Suppose next that the measure μ is not dyadically doubling, and we would like to find the class of measures μ for which $H_{\mathscr{D}}^{\mu}$ maps continuously $L^1(\mu)$ into $L^{1,\infty}(\mu)$. Characterizing the class of measures for which some operator is bounded is in general a hard problem. For instance, that is the case for the L^2 boundedness of the Cauchy integral operator in the plane and the class of linear growth measures obtained by Tolsa [22]. This led to non-standard Calderón-Zygmund theories (where μ has some polynomial growth *á la* Nazarov-Treil-Volberg and Tolsa) that one could try to apply in the present situation. This would probably require some extra (and *a posteriori* unnecessary) assumptions on μ. On the other hand, let us recall that $H_{\mathscr{D}}^{\mu}$ is a dyadic operator. Sometimes dyadic operators behave well even without assuming doubling: the dyadic Hardy-Littlewood maximal function and the dyadic square function are of weak-type $(1,1)$ for general Borel measures μ. In view of that, one could be tempted to conjecture that $H_{\mathscr{D}}^{\mu}$ is of weak-type $(1,1)$ for general measures μ without assuming any further doubling property (or polynomial growth). One could also ask the same questions for some other dyadic operators: the adjoint of the dyadic Hilbert transform, (cancellative) Haar shift operators, dyadic paraproducts or their adjoints or, more in general, non-cancellative Haar shift operators (we give

the precise definitions of these objects below). This motivates one of the main questions we address in this paper:

> *Determine the family of measures μ for which a given dyadic operator (e.g., the dyadic Hilbert transform or its adjoint, a dyadic paraproduct or its adjoint, a cancellative or non-cancellative Haar shift operator) maps continuously $L^1(\mu)$ into $L^{1,\infty}(\mu)$.*

We know already that if μ is dyadically doubling these operators satisfy weak-type estimates by a straightforward use of the standard Calderón-Zygmund theory. Therefore, it is natural to wonder whether the doubling condition is necessary or it is just convenient. As we will see along this paper there is no universal answer to that question for all the previous operators: the class of measures depends heavily on the operator in question. Let us illustrate this phenomenon with some examples:

- *Dyadic paraproducts and 1-dimensional Haar multipliers.* We shall see in Theorems 2.5, 2.11 and 5.7 that these operators are of weak-type $(1,1)$ for every locally finite Borel measure.

- *The dyadic Hilbert transform and its adjoint.* We shall prove in Theorem 2.5 that each operator gives rise to a family of measures governing the corresponding weak-type $(1,1)$. In Section 4 we shall provide some examples of measures, showing that the two classes (the one for the dyadic Hilbert transform and the one for its adjoint) are different and none of them is contained in the other. Further, the class of dyadic doubling measures is strictly contained in the intersection of the two classes.

- *Adjoints of dyadic paraproducts.* We shall obtain in Theorem 5.7 that the weak-type $(1,1)$ of these operators leads naturally to the dyadically doubling condition for μ.

Besides these examples, our main results will answer the question above providing a characterization of the measures for which any of the previous operators is of weak-type $(1,1)$. It should be pointed out that the proof of such results are relatively simple, once we have obtained the appropriate Calderón-Zygmund decomposition valid for general measures. We propose a new Calderón-Zygmund decomposition, interesting on its own right, with a new good part which will be still higher integrable. We need to do this, since the usual "good part" in the classical Calderón-Zygmund decomposition is no longer good in a general situation: the L^∞ bound (or even any higher integrability) is ruined by the fact that the average of f on a given maximal cube cannot be bounded unless the measure is assumed to be doubling or dyadically doubling. This new good part leads to an additional bad term that needs to be controlled. More precisely, fixed $\lambda > 0$, let $\{Q_j\}_j$ be the corresponding family of maximal dyadic cubes (maximal with respect to the property that $\langle |f| \rangle_Q > \lambda$, see below for notation). Then we write $f = g + b + \beta$ where

- $g \in L^p(\mu)$ for every $1 \le p < \infty$ with

$$\|g\|_{L^p(\mu)} \le C_p \, \lambda^{p-1} \, \|f\|_{L^1(\mu)};$$

- $b = \sum_j b_j$, with

$$\mathrm{supp}(b_j) \subset Q_j, \quad \int_{\mathbb{R}^d} b_j(x)\, d\mu(x) = 0, \quad \sum_j \|b_j\|_{L^1(\mu)} \le 2\, \|f\|_{L^1(\mu)};$$

- $\beta = \sum_j \beta_j$, with

$$\mathrm{supp}(\beta_j) \subset \widehat{Q}_j, \quad \int_{\mathbb{R}^d} \beta_j(x)\, d\mu(x) = 0, \quad \sum_j \|\beta_j\|_{L^1(\mu)} \le 4\, \|f\|_{L^1(\mu)}.$$

Let us compare this with the classical Calderón-Zygmund decomposition. First, we lose the L^∞ bound for the good part, however, for practical purposes this is not a problem since in most of the cases one typically uses the L^2 estimate for g. We now have two bad terms: the typical one b; and the new one β, whose building blocks are supported in the dyadic cubes $\{\widehat{Q}_j\}_j$, which are not pairwise disjoint, but still possess some cancelation. This new Calderón-Zygmund decomposition is key to obtaining the weak-type estimates for the Haar shift operators we consider.

The organization of the paper is as follows. In Section 2 we will state our main results and give some applications. Section 3 contains the proof of our main results. In Section 4 we shall present some examples of measures in $\mathbb{R}$ that are not dyadically doubling (neither have polynomial growth) for which either the dyadic Hilbert transform, its adjoint or both are of weak-type $(1,1)$. In the higher dimensional case we will review some constructions of Haar systems. We shall see that the obtained characterization depends also on the Haar system that we work with. That is, if we take a Haar shift operator (i.e., we fix the family of coefficients) and write it with different Haar systems, the conditions on the measure for the weak-type $(1,1)$ depend on the chosen Haar system. Finally, in Section 5 we present some further results including non-cancellative Haar shift operators and therefore dyadic paraproducts, and some comments about the relationship between Haar shifts and martingale transforms.

2. Main results

In this paper we study the boundedness behavior of dyadic operators with respect to Borel measures that are not necessarily doubling. For simplicity we will restrict ourselves to the Euclidean setting with the standard dyadic grid $\mathscr{D}$ in $\mathbb{R}^d$. Of course, our results should also hold for other dyadic lattices and, more in general, in the context of geometrically doubling metric spaces in terms of Christ's dyadic cubes [2], or some other dyadic constructions [6], [11]. We will use the following notation, for every $Q \in \mathscr{D}$, we let $\mathscr{D}_k(Q)$, $k \ge 1$, be the family of dyadic subcubes of side-length $2^{-k}\,\ell(Q)$. We shall work with Borel measures μ such that $\mu(Q) < \infty$ for every dyadic cube Q (equivalently, the μ-measure of every compact set is finite). To go beyond the well-known framework of the Calderón-Zygmund theory for doubling measures, the first thing we do is to develop a Calderón-Zygmund decomposition adapted to μ and to the associated dyadic maximal function

$$M_{\mathscr{D}} f(x) = \sup_{x \in Q \in \mathscr{D}} \langle |f| \rangle_Q = \sup_{x \in Q \in \mathscr{D}} \frac{1}{\mu(Q)} \int_Q |f(x)|\, d\mu(x).$$

Here we have used the notation $\langle g \rangle_Q$ for the μ-average of g on Q and we set $\langle g \rangle_Q = 0$ if $\mu(Q) = 0$. As usual, if $f \in L^1(\mu)$ and $\lambda > 0$, we cover $\{M_{\mathscr{D}}f > \lambda\}$ by the maximal dyadic cubes $\{Q_j\}_j$. In the general setting that we are considering, such maximal cubes exist (for every $\lambda > 0$) if the μ-measure of every d-dimensional quadrant is infinity. Otherwise, maximal cubes exist for λ large enough. For the sake of clarity in exposition, in the following result we assume that each d-dimensional quadrant has infinite μ-measure. The general case will be addressed in Section 3.4 below.

One could try to use the standard Calderón-Zygmund decomposition, $f = g + b$ where g and b are respectively the "good" and "bad" parts. As usual, in each Q_j the "good" part would agree with $\langle f \rangle_{Q_j}$. However, this good part would not be bounded (or even higher integrable) and therefore this decomposition would be of no use. Our new Calderón-Zygmund decomposition solves the problem with the "good" part and adds a new "bad" part whose building blocks have vanishing integrals and each of them is supported in $\widehat{Q}_j$, the dyadic parent of Q_j.

Theorem 2.1. *Let μ be a Borel measure on $\mathbb{R}^d$ satisfying that $\mu(Q) < \infty$ for all $Q \in \mathscr{D}$ and that each d-dimensional quadrant has infinite μ-measure. Given an integrable function $f \in L^1(\mu)$ and $\lambda > 0$, consider the standard covering of $\Omega_\lambda = \{M_{\mathscr{D}}f > \lambda\}$ by maximal dyadic cubes $\{Q_j\}_j$. Then we can write $f = g + b + \beta$ with*

$$g(x) = f(x)\,1_{\mathbb{R}^d \setminus \Omega_\lambda}(x) + \sum_j \langle f \rangle_{\widehat{Q}_j}\,1_{Q_j}(x)$$

$$+ \sum_j (\langle f \rangle_{Q_j} - \langle f \rangle_{\widehat{Q}_j})\frac{\mu(Q_j)}{\mu(\widehat{Q}_j)}\,1_{\widehat{Q}_j}(x),$$

$$b(x) = \sum_j b_j(x) = \sum_j (f(x) - \langle f \rangle_{Q_j})\,1_{Q_j}(x),$$

$$\beta(x) = \sum_j \beta_j(x) = \sum_j (\langle f \rangle_{Q_j} - \langle f \rangle_{\widehat{Q}_j})\left(1_{Q_j}(x) - \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)}\,1_{\widehat{Q}_j}(x)\right).$$

Then, we have the following properties:

(a) *The function g satisfies*
$$\|g\|_{L^p(\mu)}^p \leq C_p\,\lambda^{p-1}\,\|f\|_{L^1(\mu)} \quad \text{for every} \quad 1 \leq p < \infty.$$

(b) *The function b decomposes as $b = \sum_j b_j$, where*
$$\operatorname{supp}(b_j) \subset Q_j, \quad \int_{\mathbb{R}^d} b_j(x)\,d\mu(x) = 0, \quad \sum_j \|b_j\|_{L^1(\mu)} \leq 2\,\|f\|_{L^1(\mu)}.$$

(c) *The function β decomposes as $\beta = \sum_j \beta_j$, where*
$$\operatorname{supp}(\beta_j) \subset \widehat{Q}_j, \quad \int_{\mathbb{R}^d} \beta_j(x)\,d\mu(x) = 0, \quad \sum_j \|\beta_j\|_{L^1(\mu)} \leq 4\,\|f\|_{L^1(\mu)}.$$

Theorem 2.1 is closely related to Gundy's martingale decomposition [10] and was obtained in the unpublished manuscript [15] (see also [4]). It is however more flexible because the building blocks are the maximal cubes in

place of the martingale differences. This feature is crucial when considering Haar shift operators allowing us to characterize their weak-type $(1,1)$ for general Borel measures.

A baby model of the mentioned characterization —which will be illustrative for the general statement— is given by the dyadic Hilbert transform in $\mathbb{R}$ and its adjoint. To define this operator we first need to introduce some notation. First, to simplify the exposition, let us assume that $\mu(I) > 0$ for every $I \in \mathscr{D}$, below we will consider the general case. Given $I \in \mathscr{D}$ we write I_-, I_+ for the (left and right) dyadic children of I, and, as before, $\widehat{I}$ is the dyadic parent of I. We set

$$(2.2) \quad h_I = \sqrt{m(I)}\left(\frac{1_{I_-}}{\mu(I_-)} - \frac{1_{I_+}}{\mu(I_+)}\right), \qquad \text{with} \quad m(I) = \frac{\mu(I_-)\mu(I_+)}{\mu(I)}.$$

Let us first observe that the system $\mathcal{H} = \{h_I\}_{I \in \mathscr{D}}$ is orthonormal. Additionally, for every $I \in \mathscr{D}$ we have

$$(2.3) \qquad \|h_I\|_{L^1(\mu)} = 2\sqrt{m(I)}, \qquad \|h_I\|_{L^\infty(\mu)} \approx \frac{1}{\sqrt{m(I)}}.$$

Therefore we obtain the following condition which will become meaningful later

$$(2.4) \qquad \sup_{I \in \mathscr{D}} \|h_I\|_{L^\infty(\mu)}\|h_I\|_{L^1(\mu)} < \infty.$$

We define the dyadic Hilbert transform by

$$H_{\mathscr{D}}f(x) = \sum_{I \in \mathscr{D}} \langle f, h_I\rangle(h_{I_-}(x) - h_{I_+}(x)) = \sum_{I \in \mathscr{D}} \sigma(I)\langle f, h_{\widehat{I}}\rangle h_I(x),$$

where $\sigma(I) = 1$ if $I = (\widehat{I})_-$ and $\sigma(I) = -1$ if $I = (\widehat{I})_+$. Another toy model in the 1-dimensional setting is the adjoint of $H_{\mathscr{D}}$ which can be written as

$$H_{\mathscr{D}}^* f(x) = \sum_{I \in \mathscr{D}} \sigma(I)\langle f, h_I\rangle h_{\widehat{I}}(x).$$

We are going to show that the increasing or decreasing properties of m characterize the boundedness of $H_{\mathscr{D}}$ and $H_{\mathscr{D}}^*$. This motivates the following definition. We say that μ is *m-increasing* if there exists $0 < C < \infty$ such that

$$m(I) \leq C\, m(\widehat{I}), \qquad I \in \mathscr{D}.$$

We say that μ is *m-decreasing* if there exists $0 < C < \infty$ such that

$$m(\widehat{I}) \leq C\, m(I), \qquad I \in \mathscr{D}.$$

Finally, we say that μ is *m-equilibrated* if μ is both *m*-increasing and *m*-decreasing.

Let us note that if μ is the Lebesgue measure, or in general any dyadically doubling measure, we have that $m(I) \approx \mu(I)$ and therefore μ is *m*-equilibrated. As we will show below, the converse is not true. In general, we observe that

$$m(I) = \left(\frac{1}{\mu(I_-)} + \frac{1}{\mu(I_+)}\right)^{-1} \approx \min\{\mu(I_-), \mu(I_+)\} < \mu(I).$$

Thus, m gives quantitative information about the degeneracy of μ over I: $m(I)/\mu(I) \ll 1$ implies that μ mostly concentrates on only one child of I, and $m(I)/\mu(I) \gtrsim 1$ gives that $\mu(I_-) \approx \mu(I_+) \approx \mu(I)$.

We are ready to state our next result which characterizes the measures for which $H_{\mathscr{D}}$ and $H_{\mathscr{D}}^*$ are bounded for $p \neq 2$.

Theorem 2.5. *Let μ be a Borel measure on $\mathbb{R}$ satisfying that $0 < \mu(I) < \infty$ for every $I \in \mathscr{D}$.*

   (i) $H_{\mathscr{D}} : L^1(\mu) \to L^{1,\infty}(\mu)$ if and only if μ is m-increasing.

   (ii) $H_{\mathscr{D}}^* : L^1(\mu) \to L^{1,\infty}(\mu)$ if and only if μ is m-decreasing.

Moreover, if $1 < p < 2$ we have:

  (iii) $H_{\mathscr{D}} : L^p(\mu) \to L^p(\mu)$ if and only if μ is m-increasing.

  (iv) $H_{\mathscr{D}}^* : L^p(\mu) \to L^p(\mu)$ if and only if μ is m-decreasing.

If $2 < p < \infty$, by duality, the previous equivalences remain true upon switching the conditions on μ.

Furthermore, let III be a 1-dimensional Haar shift, that is, given nonnegative integers r, s,

$$(2.6) \quad \text{III}f(x) = \sum_{\substack{I \in \mathscr{D}}} \sum_{\substack{J \in \mathscr{D}_r(I) \\ K \in \mathscr{D}_s(I)}} \alpha_{J,K}^I \langle f, h_J \rangle h_K(x) \qquad with \qquad \sup_{I,J,K} |\alpha_{J,K}^I| < \infty.$$

If μ is m-equilibrated then III is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$ and from $L^p(\mu)$ to $L^p(\mu)$ for every $1 < p < \infty$.

Let us observe that our assumption on the coefficients of the Haar shift operator is not standard, below we shall explain why this is natural (see Theorem 2.11 and the comment following it).

As noted above, dyadic doubling measures are m-equilibrated. Therefore, in this case, $H_{\mathscr{D}}$, $H_{\mathscr{D}}^*$, and any 1-dimensional Haar shift III are of weak-type $(1,1)$ and bounded on $L^p(\mu)$ for every $1 < p < \infty$. In Section 4.1 we shall present examples of measures in $\mathbb{R}$ as follows:

- μ is m-equilibrated, but μ is not dyadically doubling, not of polynomial growth. Thus, we have an example of a measure that is out of the classical theory for which the dyadic Hilbert transform, its adjoint and any Haar shift is of weak-type $(1,1)$ and bounded on $L^p(\mu)$ for every $1 < p < \infty$.

- μ is m-increasing, but μ is not m-decreasing, not dyadically doubling, not of polynomial growth. Thus, $H_{\mathscr{D}}$ is of weak-type $(1,1)$, bounded on $L^p(\mu)$ for every $1 < p \leq 2$ and unbounded on $L^p(\mu)$ for $2 < p < \infty$; $H_{\mathscr{D}}^*$ is bounded on $L^p(\mu)$ for $2 \leq p < \infty$, not of weak-type $(1,1)$ and unbounded on $L^p(\mu)$ for every $1 < p < 2$.

- μ is m-decreasing, but μ is not m-increasing, not dyadically doubling, not of polynomial growth. Thus, $H_{\mathscr{D}}$ is bounded on $L^p(\mu)$ for $2 \leq p < \infty$, not of weak-type $(1,1)$ and unbounded on $L^p(\mu)$ for every $1 < p < 2$; $H_{\mathscr{D}}^*$ is of weak-type $(1,1)$, bounded on $L^p(\mu)$ for every $1 < p \leq 2$ and unbounded on $L^p(\mu)$ for $2 < p < \infty$.

- μ is not m-decreasing, not m-increasing, not dyadically doubling, but μ has polynomial growth. Thus, this is an example of a measure *á la* Nazarov-Treil-Volberg and Tolsa for which $H_{\mathscr{D}}$ and $H_{\mathscr{D}}^*$ are bounded on $L^2(\mu)$, unbounded on $L^p(\mu)$ for $1 < p < \infty$, $p \neq 2$, and not of weak-type $(1,1)$.

Our next goal is to extend the previous result to higher dimensions. In this case we do not necessarily assume that the measures have full support. The building blocks, that is, the Haar functions are no in one-to-one correspondence to the dyadic cubes: associated to every cube Q we expect to have at most $2^d - 1$ linearly independent Haar functions. Moreover, there are different ways to construct a Haar system (see Section 4.2 below). We next define the Haar systems that we are going to use:

Definition 2.7. Let μ be a Borel measure on $\mathbb{R}^d$, $d \geq 1$, satisfying that $\mu(Q) < \infty$ for every $Q \in \mathscr{D}$. We say that $\Phi = \{\phi_Q\}_{Q \in \mathscr{D}}$ is a *generalized Haar system* in $\mathbb{R}^d$ if the following conditions hold:

(a) For every $Q \in \mathscr{D}$, $\operatorname{supp}(\phi_Q) \subset Q$.

(b) If Q', $Q \in \mathscr{D}$ and $Q' \subsetneq Q$, then ϕ_Q is constant on Q'.

(c) For every $Q \in \mathscr{D}$, $\displaystyle\int_{\mathbb{R}^d} \phi_Q(x)\, d\mu(x) = 0$.

(d) For every $Q \in \mathscr{D}$, either $\|\phi_Q\|_{L^2(\mu)} = 1$ or $\phi_Q \equiv 0$.

Remark 2.8. The following comments pertain to the previous definition.

- Given a generalized Haar system $\Phi = \{\phi_Q\}_{Q \in \mathscr{D}}$, we write $\mathscr{D}_\Phi$ for the set of dyadic cubes Q for which $\phi_Q \not\equiv 0$. By assumption, we allow $\mathscr{D}_\Phi$ to be a proper subcollection of $\mathscr{D}$. Note that $\{\phi_Q\}_{Q \in \mathscr{D}}$ is an orthogonal system whereas $\{\phi_Q\}_{Q \in \mathscr{D}_\Phi}$ is orthonormal.

  Let us point out that we allow the measure μ to vanish in some dyadic cubes. If $\mu(Q) = 0$, we must have $\phi_Q \equiv 0$ and therefore $Q \in \mathscr{D} \setminus \mathscr{D}_\Phi$. If $\mu(Q) = \mu(Q')$ for some child Q' of Q (i.e., every brother of Q' has null μ-measure) then $\phi_Q \equiv 0$ and thus $Q \in \mathscr{D} \setminus \mathscr{D}_\Phi$. Suppose now that $Q \in \mathscr{D}_\Phi$ (therefore $\mu(Q) > 0$), by convention, we set $\phi_Q \equiv 0$ in every dyadic child of Q with vanishing measure.

- Let us suppose that for every $Q \in \mathscr{D}_\Phi$, ϕ_Q takes exactly 2 different non-zero values (call Φ a *2-value* generalized Haar system). In view of the previous remark, ϕ_Q is "uniquely" determined modulo a multiplicative ± 1. That is, we can find E_Q^+, $E_Q^- \subset Q$, such that $E_Q^+ \cap E_Q^- = \emptyset$, $E_Q^\pm$ is comprised of dyadic children of Q, $\mu(E_Q^\pm) > 0$ and

$$(2.9) \quad \phi_Q = \sqrt{m_\Phi(Q)} \left(\frac{1_{E_Q^-}}{\mu(E_Q^-)} - \frac{1_{E_Q^+}}{\mu(E_Q^+)} \right), \qquad m_\Phi(Q) = \frac{\mu(E_Q^-)\mu(E_Q^+)}{\mu(E_Q^- \cup E_Q^+)}.$$

  Then, for every $Q \in \mathscr{D}_\Phi$ we have

$$(2.10) \qquad \|\phi_Q\|_{L^1(\mu)} = 2\sqrt{m_\Phi(Q)}, \qquad \|\phi_Q\|_{L^\infty(\mu)} \approx \frac{1}{\sqrt{m_\Phi(Q)}}.$$

- In dimension 1, if we assume as before that $\mu(I) > 0$ for every $I \in \mathscr{D}$, we then have that $\mathcal{H}$ defined above is a generalized Haar system in $\mathbb{R}$ with $\mathscr{D}_{\mathcal{H}} = \mathscr{D}$. The previous remark and the fact every dyadic interval has two children say that $\mathcal{H}$ is "unique" in the following sense: let Φ be a generalized Haar system in $\mathbb{R}$, then $\phi_I = \pm h_I$ for every $I \in \mathscr{D}_\Phi$. Note that we can now allow the measure to vanish on some dyadic cubes. In such a case we will have that $\phi_I \equiv 0$ for every $I \in \mathscr{D}$ for which $\mu(I_-) \cdot \mu(I_+) = 0$. Also, $\phi_I = \pm h_I$ and $m_\Phi(I) = m(I)$ for every $I \in \mathscr{D}_\Phi$.

Our main result concerning general Haar shift operators characterizes the weak-type $(1,1)$ in terms of the measure μ and the generalized Haar systems that define the operator. In Section 5.1 we shall also consider non-cancellative Haar shift operators where condition (c) in Definition 2.7 is dropped for the Haar systems Φ and Ψ. This will allow us to obtain similar results for dyadic paraproducts.

Theorem 2.11. *Let μ be a Borel measure on $\mathbb{R}^d$, $d \geq 1$, such that $\mu(Q) < \infty$ for every $Q \in \mathscr{D}$. Let $\Phi = \{\phi_Q\}_{Q \in \mathscr{D}}$ and $\Psi = \{\psi_Q\}_{Q \in \mathscr{D}}$ be two generalized Haar systems in $\mathbb{R}^d$. Given two non-negative integers r, s we set*

$$\Xi(\Phi, \Psi; r, s) = \sup_{Q \in \mathscr{D}} \left\{ \|\phi_R\|_{L^\infty(\mu)} \|\psi_S\|_{L^1(\mu)} : R \in \mathscr{D}_r(Q), S \in \mathscr{D}_s(Q) \right\}.$$

Let $\text{Ш}_{r,s}$ be a Haar shift of complexity (r,s), that is,

$$\text{Ш}_{r,s} f(x) = \sum_{Q \in \mathscr{D}} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \alpha_{R,S}^Q \langle f, \phi_R \rangle \psi_S(x) \qquad with \qquad \sup_{Q,R,S} |\alpha_{R,S}^Q| < \infty.$$

If $\Xi(\Phi, \Psi; r, s) < \infty$, then $\text{Ш}_{r,s}$ maps continuously $L^1(\mu)$ into $L^{1,\infty}(\mu)$, and by interpolation $\text{Ш}_{r,s}$ is bounded on $L^p(\mu)$, $1 < p \leq 2$.

Conversely, let $\text{Ш}_{r,s}$ be a Haar shift of complexity (r,s) satisfying the non-degeneracy condition $\inf_{Q,R,S} |\alpha_{R,S}^Q| > 0$. If $\text{Ш}_{r,s}$ maps continuously $L^1(\mu)$ into $L^{1,\infty}(\mu)$ then $\Xi(\Phi, \Psi; r, s) < \infty$.

Let us point out that in the Euclidean setting with the Lebesgue measure one typically assumes that $|\alpha_{R,S}^Q| \lesssim (|R||S|)^{1/2}/|Q|$. Our condition, with a general measure, is less restrictive and more natural: having assumed the corresponding condition with respect to μ, $H_\mathscr{D}$ and $H_\mathscr{D}^*$ would not be 1-dimensional Haar shift operators unless μ is dyadically doubling.

To illustrate the generality and the applicability of Theorem 2.11 we consider some examples. Before doing that we need to introduce some notation. Let Φ be a generalized Haar system in $\mathbb{R}^d$, we say that Φ is *standard* if

$$(2.12) \qquad \sup_{Q \in \mathscr{D}} \|\phi_Q\|_{L^1(\mu)} \|\phi_Q\|_{L^\infty(\mu)} < \infty.$$

Note that we can restrict the supremum to $Q \in \mathscr{D}_\Phi$. Also, if $Q \in \mathscr{D}_\Phi$, Hölder's inequality and (d) imply that each term in the supremum is bounded from below by 1. Thus, Φ being standard says that the previous quantity is bounded from below and from above uniformly for every $Q \in \mathscr{D}_\Phi$. Notice that in the language of Theorem 2.11, Φ being standard is equivalent to $\Xi(\Phi, \Phi; 0, 0) < \infty$.

Remark 2.13. If Φ is a 2-value generalized Haar system, (2.10) implies that Φ is standard. Note that in $\mathbb{R}$ (since every dyadic interval has two children) every generalized Haar system, including $\mathcal{H}$ introduced above, is of 2-value type and therefore standard.

Example 2.14 (Haar multipliers). Let $\Phi = \{\phi_Q\}_Q$ be a generalized Haar system in $\mathbb{R}^d$. We take the Haar shift operator of complexity $(r, s) = (0, 0)$, usually referred to as a Haar multiplier,

$$\text{Ш}_{0,0}f(x) = \sum_{Q \in \mathscr{D}} \alpha_Q \langle f, \phi_Q \rangle \phi_Q(x), \qquad \text{with} \quad \sup_Q |\alpha_Q| < \infty.$$

Then $\Xi(\Phi, \Phi; 0, 0) < \infty$ is equivalent to the fact that Φ is standard. Therefore Theorem 2.11 says that $\text{Ш}_{0,0}$ is of weak-type $(1, 1)$ provided Φ is standard. We also have the converse for non-degenerate Haar shifts of complexity $(0, 0)$. As a consequence of these we have the following characterization: "Φ is standard if and only if all Haar multipliers are of weak-type $(1, 1)$". As observed above this can be applied to any 2-value generalized Haar system in $\mathbb{R}^d$. In particular, for an arbitrary measure in $\mathbb{R}$ such that $\mu(I) > 0$ for every $I \in \mathscr{D}$, all Haar multipliers of the form

$$\text{Ш}_{0,0}f(x) = \sum_{I \in \mathscr{D}} \alpha_I \langle f, h_I \rangle h_I(x), \qquad \text{with} \quad \sup_I |\alpha_I| < \infty,$$

are of weak-type $(1, 1)$. In higher dimensions, taking an arbitrary measure such that $\mu(Q) > 0$ for every $Q \in \mathscr{D}$, any Haar multiplier as above defined in terms of a 2-value generalized Haar system in $\mathbb{R}^d$ is of weak-type $(1, 1)$. We note that we cannot remove the assumption that the system is 2-value: in Section 4.2 we shall give an example of a generalized Haar system that is not standard and a Haar multiplier that is not of weak-type $(1, 1)$. All these comments can be generalized to measures without full support.

Example 2.15 (The dyadic Hilbert transform I). For simplicity, we first suppose that $\mu(I) > 0$ for every $I \in \mathscr{D}$. The dyadic Hilbert transform in $\mathbb{R}$ can be seen as the non-degenerate Haar shift $H_{\mathscr{D}} = \text{Ш}_{0,1}$ with $\alpha^I_{I,I_\pm} = \mp 1$. Theorem 2.11 says that $H_{\mathscr{D}}$ is of weak-type $(1, 1)$ if and only if $\Xi(\mathcal{H}, \mathcal{H}; 0, 1) < \infty$, which in view of (2.3) is equivalent to the fact that μ is m-increasing. For the adjoint of the dyadic Hilbert transform $H^*_{\mathscr{D}} = \text{Ш}_{1,0}$ with $\alpha^I_{I_\pm, I} = \mp 1$ and this is a non-degenerate Haar shift. Again, Theorem 2.11 characterizes the weak-type $(1, 1)$ of $H^*_{\mathscr{D}}$ in terms of $\Xi(\mathcal{H}, \mathcal{H}; 1, 0) < \infty$, which this time rewrites into the property that μ is m-decreasing.

Example 2.16 (The dyadic Hilbert transform II). We now consider the dyadic Hilbert transform but with respect to measures that may vanish. Let Φ be a generalized Haar system in $\mathbb{R}$ and let $\mathscr{D}_\Phi$ be as before. By the discussion above we may suppose that $\phi_I = h_I$ for every $I \in \mathscr{D}_\Phi$. Then, the corresponding dyadic Hilbert transform can be written as

$$H_{\mathscr{D}, \Phi}f = \sum_{I \in \mathscr{D}} \langle f, \phi_I \rangle (\phi_{I_-} - \phi_{I_+}) = \sum_{I \in \mathscr{D}_\Phi : \widehat{I} \in \mathscr{D}_\Phi} \sigma(I) \langle f, h_{\widehat{I}} \rangle h_I.$$

As before we have that $H_{\mathscr{D}, \Phi} = \text{Ш}_{0,1}$ is non-degenerate. Therefore its weak-type $(1, 1)$ is characterized in terms of the finiteness of $\Xi(\Phi, \Phi; 0, 1)$. Thus,

we obtain that

$$H_{\mathscr{D},\Phi} : L^1(\mu) \longrightarrow L^{1,\infty}(\mu) \quad \Longleftrightarrow \quad m(I) \leq C\, m(\widehat{I}), \; I, \widehat{I} \in \mathscr{D}_\Phi.$$

Note that the latter condition says that μ is m-increasing on the family $\mathscr{D}_\Phi$ (so in particular the intervals with zero μ-measure or those with one child of zero μ-measure do not count).

For the adjoint of $H_{\mathscr{D},\Phi}$ we have

$$H_{\mathscr{D},\Phi}^* f(x) = \sum_{I \in \mathscr{D}} \sigma(I)\langle f, \phi_I \rangle \phi_{\widehat{I}} = \sum_{I \in \mathscr{D}_\Phi : \widehat{I} \in \mathscr{D}_\Phi} \sigma(I)\langle f, h_I \rangle h_{\widehat{I}}$$

and we can analogously obtain

$$H_{\mathscr{D},\Phi}^* : L^1(\mu) \longrightarrow L^{1,\infty}(\mu) \quad \Longleftrightarrow \quad m(\widehat{I}) \leq C\, m(I), \; I, \widehat{I} \in \mathscr{D}_\Phi.$$

Example 2.17 (Haar Shifts in $\mathbb{R}$). We start with the case $\mu(I) > 0$ for every $I \in \mathscr{D}$. Let us consider $\text{Ш} = \text{Ш}_{r,s}$ as in (2.6), that is, a Haar shift operator of complexity (r, s) defined in terms of the system $\mathcal{H}$. By Theorem 2.11 we know that $\Xi(\mathcal{H}, \mathcal{H}; r, s) < \infty$ is sufficient (and necessary if we knew that Ш is non-degenerate) for the weak-type $(1, 1)$. We can rewrite this condition as follows: $m(K) \lesssim m(J)$ for every $I \in \mathscr{D}$, $J \in \mathscr{D}_r(I)$, $K \in \mathscr{D}_s(I)$. If μ is m-equilibrated then $m(J) \approx m(I)$ and $m(K) \approx m(I)$ for every $I \in \mathscr{D}$, $J \in \mathscr{D}_r(I)$, $K \in \mathscr{D}_s(I)$. All these and (2.4) give at once $\Xi(\mathcal{H}, \mathcal{H}; r, s) < \infty$ for every $r, s \geq 0$. Thus, in dimension 1, the fact μ is m-equilibrated implies that every Haar shift operator is of weak-type $(1, 1)$. We would like to recall that in Section 4 we shall construct measures that are m-equilibrated but are neither dyadically doubling nor of polynomial growth. Thus, Haar shift operators are a large family of (dyadic) Calderón-Zygmund operators obeying a weak-type $(1, 1)$ bound with underlaying measures that do not satisfy those classical conditions.

For measures vanishing in some cubes, Theorem 2.11 gives us a sufficient (and often necessary) condition. However, it is not clear whether in such a case one can write that condition in terms of μ being m-equilibrated. We would need to be able to compare $m(K)$ and $m(J)$ for K and J as before with the additional condition that $J, K \in \mathscr{D}_\Phi$. Note that the fact that μ is m-equilibrated gives information about jumps of order 1 in the generations and it could happen that we cannot "connect" J and K with "1-jumps" within $\mathscr{D}_\Phi$. Take for instance $I = [0, 1)$, $J = [0, 4)$, $d\mu(x) = 1_{[0,1)\cup[2,4)}(x)\, dx$, $\Phi = \{h_I, h_J\}$ and $\text{Ш}_{2,0} = \langle f, h_I \rangle h_J$. Then Theorem 2.11 says that $\text{Ш}_{2,0}$ is of weak-type $(1, 1)$ since $\Xi(\Phi, \Phi; 2, 0) = 4\,(m[0, 4) \cdot m[0, 1))^{1/2} = 4/\sqrt{6} < \infty$. However, $\mathscr{D}_\Phi = \{I, J\}$ and these two dyadic intervals are 2-generation separated.

Example 2.18 (Haar Shifts in $\mathbb{R}^d$ for 2-value generalized Haar systems). Let us suppose that Φ and Ψ are 2-value generalized Haar systems. Write $E_Q^\pm$ (resp. $F_Q^\pm$) for the sets associated with $\phi_Q \in \mathscr{D}_\Phi$ (resp. $\psi_Q \in \mathscr{D}_\Psi$), see (2.9). By (2.10) we have that $\Xi(\Phi, \Psi; r, s) < \infty$ if an only if μ satisfies

$$(2.19) \qquad m_\Psi(S) = \frac{\mu(F_S^-)\mu(F_S^+)}{\mu(F_S^- \cup F_S^+)} \lesssim \frac{\mu(E_R^-)\mu(E_R^+)}{\mu(E_Q^- \cup E_R^+)} = m_\Phi(R)$$

for every $Q \in \mathscr{D}$, $R \in \mathscr{D}_r(Q)$, $S \in \mathscr{D}_s(Q)$, $R \in \mathscr{D}_\Phi$ and $S \in \mathscr{D}_\Psi$. Therefore Theorem 2.11 says that $\text{III}_{r,s}$ is of weak-type $(1,1)$ provided μ satisfies the condition (2.19). The converse holds provided $\text{III}_{r,s}$ is non-degenerated.

3. Proofs of the main results

3.1. **A new Calderón-Zygmund decomposition.** As pointed out before, we shall work with the standard dyadic filtration $\mathscr{D} = \bigcup_{k \in \mathbb{Z}} \mathscr{D}_k$ in $\mathbb{R}^d$, but all our results hold for any other dyadic lattice. If $k \geq 0$ is a nonnegative integer, we write $\mathscr{D}_k(Q)$ for the partition of Q into dyadic subcubes of side-length $2^{-k} \ell(Q)$ and $Q^{(k)}$ for its k-th dyadic ancestor, i.e., the only cube of side-length $2^k \ell(Q)$ that contains Q. The cubes in $\mathscr{D}_1(Q)$ are called dyadic children of Q and $\widehat{Q} = Q^{(1)}$ is the dyadic parent of Q.

By μ we will denote any positive Borel measure on $\mathbb{R}^d$ such that $\mu(Q) < \infty$ for all $Q \in \mathscr{D}$. Write $\mathcal{M}$ for the class of such measures. Once μ is fixed, we set for $Q \in \mathscr{D}$

$$\langle f \rangle_Q = \frac{1}{\mu(Q)} \int_Q f(x) \, d\mu(x) \qquad \text{with} \quad \langle f \rangle_Q = 0 \ \text{ when } \ \mu(Q) = 0.$$

The dyadic maximal operator for $\mu \in \mathcal{M}$ is then $M_{\mathscr{D}} f(x) = \sup_{x \in Q \in \mathscr{D}} \langle |f| \rangle_Q$.

Let us write $\mathbb{R}^d_j$, $1 \leq j \leq 2^d$, for the d-dimensional quadrants in $\mathbb{R}^d$. It will be convenient to consider temporarily the subclass $\mathcal{M}_\infty$ of measures $\mu \in \mathcal{M}$ such that $\mu(\mathbb{R}^d_j) = \infty$ for all $1 \leq j \leq 2^d$. We will prove our main results under the assumption that $\mu \in \mathcal{M}_\infty$ and sketch in Section 3.4 the modifications needed to adapt our arguments for any $\mu \in \mathcal{M}$.

Assuming now that $\mu \in \mathcal{M}_\infty$, we know that $\langle |f| \rangle_Q \to 0$ as $\ell(Q) \to \infty$ whenever $f \in L^1(\mu)$. In particular, given any $\lambda > 0$, there exists a collection of disjoint maximal dyadic cubes $\{Q_j\}_j$ such that

$$\Omega_\lambda = \left\{ x \in \mathbb{R}^d : \ M_{\mathscr{D}} f(x) > \lambda \right\} = \bigcup_j Q_j,$$

where the cubes $\{Q_j\}_j$ are maximal in the sense that for all dyadic cubes $Q \supsetneq Q_j$ we have

$$(3.1) \qquad\qquad \langle |f| \rangle_Q \leq \lambda < \langle |f| \rangle_{Q_j},$$

Using this covering of the level set Ω_λ, we can reproduce the classical estimate to show the weak-type $(1,1)$ boundedness of the dyadic Hardy-Littlewood maximal operator. Note that maximal cubes have positive measure by construction.

Proof of Theorem 2.1. We are currently assuming that $\mu \in \mathcal{M}_\infty$, see Section 3.4 for the modifications needed in the general case. By construction, $f = g + b + \beta$. Moreover, the support and mean-zero conditions for b_j and β_j can be easily checked. On the other hand, since the cubes Q_j are pairwise disjoint

$$\sum_j \|b_j\|_{L^1(\mu)} \leq 2 \sum_j \int_{Q_j} |f(x)| \, d\mu(x) \leq 2 \|f\|_{L^1(\mu)}.$$

Similarly, by the maximality of the Calderón-Zygmund cubes, see (3.1), we obtain

$$\sum_j \|\beta_j\|_{L^1(\mu)} \le \sum_j 2(\langle|f|\rangle_{Q_j} + \langle|f|\rangle_{\widehat{Q}_j})\mu(Q_j) \le 4\sum_j \int_{Q_j} |f|\,d\mu \le 4\|f\|_{L^1(\mu)}.$$

It remains to prove the norm inequalities for g. Write g_1, g_2 and g_3 for each of the terms defining g and let us estimate these in turn. It is immediate that $\|g_1\|_{L^1(\mu)} \le \|f\|_{L^1(\mu)}$. Since $M_{\mathscr{D}}$ is of weak-type $(1,1)$, Lebesgue's differentiation theorem yields $\|g_1\|_{L^\infty(\mu)} \le \|M_{\mathscr{D}}f \cdot 1_{\mathbb{R}^d\setminus\Omega_\lambda}\|_{L^\infty(\mu)} \le \lambda$. The estimates for g_2 are similar. Since $\langle|f|\rangle_{\widehat{Q}_j} \le \lambda$, we obtain

$$\|g_2\|_{L^1(\mu)} \le \lambda\,\mu(\Omega_\lambda) \le \|f\|_{L^1(\mu)} \qquad \text{and} \qquad \|g_2\|_{L^\infty(\mu)} \le \lambda.$$

These estimates immediately yield the corresponding $L^p(\mu)$-estimates for g_1 and g_2.

The estimate for g_3 is not straightforward: each term in the sum is supported in $\widehat{Q}_j$, and these sets are not pairwise disjoint in general. In particular, an L^∞ estimate is not to be expected. However, we do have that

$$|g_3(x)| \le \sum_j \left(\langle|f|\rangle_{Q_j} + \langle|f|\rangle_{\widehat{Q}_j}\right)\frac{\mu(Q_j)}{\mu(\widehat{Q}_j)}\,1_{\widehat{Q}_j}(x)$$

$$\le 2\sum_j \left(\int_{Q_j} |f(y)|\,d\mu(y)\right)\frac{1}{\mu(\widehat{Q}_j)}\,1_{\widehat{Q}_j}(x) =: 2\,Tf(x).$$

The following lemma contains the relevant estimates for T:

Lemma 3.2. *Let $\{Q_j\}$ be a family of pairwise disjoint dyadic cubes and set*

$$Tf(x) = \sum_j \left(\int_{Q_j} |f(y)|\,d\mu(y)\right)\frac{1}{\mu(\widehat{Q}_j)}\,1_{\widehat{Q}_j}(x).$$

For every $m \in \mathbb{N}$, T satisfies the estimate

$$\|Tf\|_{L^m(\mu)}^m \le m!\left(\sup_j \frac{1}{\mu(\widehat{Q}_j)}\int_{\widehat{Q}_j} |f(y)|\,d\mu(y)\right)^{m-1}\int_{\bigcup_j Q_j} |f(x)|\,d\mu(x)$$

Assume this result momentarily. The case $m = 1$ implies $\|g_3\|_{L^1(\mu)} \le 2\,\|f\|_{L^1(\mu)}$. On the other hand, applying it for a general integer m, we get by (3.1)

$$\|g_3\|_{L^m(\mu)}^m \le 2^m\,m!\,\lambda^{m-1}\,\|f\|_{L^1(\mu)}.$$

Now, if $1 < p < \infty$ is not an integer, we take $m = [p] + 1$ and let $0 < \theta < 1$ be such that $p = \theta + (1-\theta)m$. Then, by Hölder's inequality with indices $\frac{1}{\theta}$ and $\frac{1}{1-\theta}$, we obtain as desired

$$\|g_3\|_{L^p(\mu)}^p \le \|g_3\|_{L^1(\mu)}^\theta \|g_3\|_{L^m(\mu)}^{(1-\theta)m} \le 2^p(m!)^{\frac{p-1}{m-1}}\lambda^{p-1}\|f\|_{L^1(\mu)}.$$

$\square$

Proof of Lemma 3.2. The case $m = 1$ is trivial. Let us proceed by induction and assume that the estimate for m holds. Write $\varphi_j = \frac{1}{\mu(\widehat{Q}_j)}\int_{Q_j} |f|\,d\mu$ and define the sets

$$\Lambda_k = \left\{(j_1, j_2, \ldots, j_{m+1}) \in \mathbb{N}^{m+1} : \widehat{Q}_{j_k} = \widehat{Q}_{j_1} \cap \widehat{Q}_{j_2} \cap \cdots \cap \widehat{Q}_{j_{m+1}}\right\}.$$

By symmetry we obtain

$$
\begin{aligned}
\|Tf\|_{L^{m+1}(\mu)}^{m+1} &\leq \sum_{k=1}^{m+1} \sum_{\Lambda_k} \varphi_{j_1} \cdots \varphi_{j_{m+1}} \mu(\widehat{Q}_{j_1} \cap \cdots \cap \widehat{Q}_{j_{m+1}}) \\
&= (m+1) \sum_{\Lambda_{m+1}} \varphi_{j_1} \cdots \varphi_{j_m} \int_{Q_{j_{m+1}}} |f(x)| \, d\mu(x) \\
&\leq (m+1) \sum_{j_1,\ldots,j_m} \varphi_{j_1} \cdots \varphi_{j_m} \int_{\widehat{Q}_{j_1} \cap \cdots \cap \widehat{Q}_{j_m}} |f(x)| \, d\mu(x) \\
&\leq (m+1) \Big(\sup_j \frac{1}{\mu(\widehat{Q}_j)} \int_{\widehat{Q}_j} |f| \, d\mu \Big) \sum_{j_1,\ldots,j_m} \varphi_{j_1} \cdots \varphi_{j_m} \mu(\widehat{Q}_{j_1} \cap \cdots \cap \widehat{Q}_{j_m}) \\
&= (m+1) \Big(\sup_j \frac{1}{\mu(\widehat{Q}_j)} \int_{\widehat{Q}_j} |f| \, d\mu \Big) \|Tf\|_{L^m(\mu)}^m.
\end{aligned}
$$

Note that the second inequality uses that the cubes $\{Q_j\}_j$ are pairwise disjoint, and the third inequality uses that $\widehat{Q}_{j_1} \cap \cdots \cap \widehat{Q}_{j_m} = \widehat{Q}_{j_i}$ for some $1 \leq i \leq m$. This and the induction hypothesis yield at once the desired estimate and the proof is complete. $\qquad\square$

The new Calderón-Zygmund decomposition in Theorem 2.1 can be used to obtain that some classical operators are of weak-type $(1,1)$ for general Borel measures: the ℓ^q-valued dyadic Hardy-Littlewood maximal function with $1 < q < \infty$, the dyadic square function, and 1-dimensional Haar multipliers. For the first operator, one needs a straightforward sequence-valued extension of the new Calderón-Zygmund decomposition and the reader is referred to [4]. Let us then look at the dyadic square function

$$
\mathcal{S}f(x) = \Big(\sum_{Q \in \mathscr{D}} |\langle f \rangle_Q - \langle f \rangle_{\widehat{Q}}|^2 1_Q(x) \Big)^{1/2}.
$$

It is well-known that $\mathcal{S}$ is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$ with a proof adopting a probabilistic point of view. However, using our Calderón-Zygmund decomposition one can reprove this result using harmonic analysis techniques as follows. We decompose $f = g + b + \beta$ as in Theorem 2.1. The estimate for the good part is standard using that $\mathcal{S}$ is bounded on $L^2(\mu)$ and (a) in Theorem 2.1. For the bad terms, using the weak-type $(1,1)$ of $M_{\mathscr{D}}$, it suffices to restrict the level set to $\mathbb{R}^d \setminus \Omega_\lambda$. Theorem 2.1 parts (b) and (c) yield respectively that $(\mathcal{S}b_j) 1_{\mathbb{R}^d \setminus Q_j} \equiv 0$ and $(\mathcal{S}\beta_j) 1_{\mathbb{R}^d \setminus \widehat{Q}_j} \equiv 0$. Thus everything is reduced to the following

$$
\mu\{x \in \mathbb{R}^d \setminus \Omega_\lambda : \mathcal{S}\beta(x) > \lambda/2\} \leq \frac{2}{\lambda} \sum_j \int_{\widehat{Q}_j \setminus Q_j} |\mathcal{S}\beta_j| \, d\mu
$$

$$
= \frac{2}{\lambda} \sum_j |\langle f \rangle_{Q_j} - \langle f \rangle_{\widehat{Q}_j}| \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)} \mu(\widehat{Q}_j \setminus Q_j) \leq \frac{4}{\lambda} \sum_j \int_{Q_j} |f| \, d\mu \leq \frac{4}{\lambda} \|f\|_{L^1(\mu)}.
$$

All these ingredients allow one to conclude that $\mathcal{S}$ is of weak-type $(1,1)$. Details are left to the reader

Finally, under the assumption that $0 < \mu(I) < \infty$ for all $I \in \mathscr{D}$, we consider the 1-dimensional Haar multipliers defined as

$$T_\alpha f(x) = \sum_{I \in \mathscr{D}} \alpha_I \langle f, h_I \rangle h_I(x), \qquad \sup_I |\alpha_I| < \infty.$$

This operator is bounded on $L^2(\mu)$ by orthonormality. A probabilistic point of view, see Section 5.3, yields that T_α is a dyadic martingale transform and therefore of weak-type $(1,1)$. Again, our new decomposition gives a proof with a "harmonic analysis" flavor. We first observe that $T_\alpha b_j(x) = 0$ for every $x \in \mathbb{R} \setminus Q_j$. Therefore, using Theorem 2.1 and proceeding as above everything reduces to the following estimate

$$\mu\{x \in \mathbb{R} : |T_\alpha \beta(x)| > \lambda/2\} \leq \frac{2}{\lambda} \sum_j |\alpha_{\widehat{I}_j}| \, |\langle f \rangle_{I_j} - \langle f \rangle_{\widehat{I}_j}| \sqrt{m(\widehat{I}_j)} \|h_{\widehat{I}_j}\|_{L^1(\mu)}$$

$$\leq \sup_I |\alpha_I| \frac{8}{\lambda} \sum_j \langle |f| \rangle_{I_j} m(\widehat{I}_j) \leq \sup_I |\alpha_I| \frac{8}{\lambda} \|f\|_{L^1(\mu)},$$

where we have used (3.4) below, (3.1), (2.3) and that $m(\widehat{I}_j) < \mu(I_j)$.

3.2. **The dyadic Hilbert transform.** In this section we prove Theorem 2.5. Although the estimates for $H_\mathscr{D}$ and $H_\mathscr{D}^*$ follow from Theorem 2.11 as explained above, we believe that it is worth giving the argument: the proofs for our toy models $H_\mathscr{D}$ and $H_\mathscr{D}^*$ are much simpler and have motivated our general result. We will skip, however, the last statement in the result since it follows from Theorem 2.11, as explained in Example 2.17, and interpolation.

Before starting the proof we observe that by the orthonormality of the system $\mathcal{H}$ we have

$$(3.3) \qquad \|H_\mathscr{D} f\|_{L^2(\mu)}^2 = \sum_{I \in \mathscr{D}} |\langle f, h_{\widehat{I}} \rangle|^2 \leq 2\|f\|_{L^2(\mu)}^2.$$

Thus, $H_\mathscr{D}$ and $H_\mathscr{D}^*$ are bounded on $L^2(\mu)$.

Proof of Theorem 2.5, part (i). We first prove the necessity of μ being m-increasing. Take $f = h_I$ so that $H_\mathscr{D} f = h_{I_-} - h_{I_+}$. Using that h_I is constant on dyadic subintervals of I and that $H_\mathscr{D}$ is of weak-type $(1,1)$ we obtain that μ is m-increasing:

$$\left(\sqrt{m(I_-)} + \sqrt{m(I_+)} \right) \approx \|h_{I_-}\|_{L^1(\mu)} + \|h_{I_+}\|_{L^1(\mu)}$$

$$\approx \|H_\mathscr{D} h_I\|_{L^{1,\infty}(\mu)} \lesssim \|h_I\|_{L^1(\mu)} \approx \sqrt{m(I)}.$$

Next we obtain that if μ is m-increasing then $H_\mathscr{D}$ is of weak-type $(1,1)$. In order to use Theorem 2.1, we shall assume that $\mu \in \mathcal{M}_\infty$, that is, $\mu[0,\infty) = \mu(-\infty,0) = \infty$. The general case will be considered in Section 3.4 below. Fix $\lambda > 0$ and decompose f by means of the Calderón-Zygmund decomposition in Theorem 2.1. Hence,

$$\mu\{x \in \mathbb{R} : |H_\mathscr{D} f(x)| > \lambda\} \leq \mu\{x \in \mathbb{R} : |H_\mathscr{D} g(x)| > \lambda/3\} + \mu(\Omega_\lambda)$$

$$+ \mu\{x \in \mathbb{R} \setminus \Omega_\lambda : |H_\mathscr{D} b(x)| > \lambda/3\} + \mu\{x \in \mathbb{R} : |H_\mathscr{D} \beta(x)| > \lambda/3\}$$

$$= S_1 + S_2 + S_3 + S_4.$$

Using the weak-type $(1,1)$ for $M_{\mathscr{D}}$, Theorem 2.1 part (a) and (3.3) it is standard to check that $S_1 + S_2 \leq (C/\lambda)\|f\|_{L^1(\mu)}$. Using that each b_j has vanishing integral and that h_I is constant in each $I_{\pm}$ it is easy to see that $H_{\mathscr{D}}b_j(x) = 0$ whenever $x \in \mathbb{R} \setminus I_j$ and thus $S_3 = 0$. To estimate S_4 we first observe that

$$(3.4) \qquad \langle \beta_j, h_I \rangle = \sigma(I_j)(\langle f \rangle_{I_j} - \langle f \rangle_{\widehat{I}_j})\sqrt{m(\widehat{I}_j)}\,\delta_{\widehat{I}_j, I}.$$

This can be easily obtained using that β_j and h_I have vanishing integral; that β_j is supported on $\widehat{I}_j$ and constant in each dyadic children of $\widehat{I}_j$; and that h_I is supported on I. Thus,

$$H_{\mathscr{D}}\beta_j = \sum_{I \in \mathscr{D}} \sigma(I)\langle \beta_j, h_{\widehat{I}}\rangle h_I = (\langle f \rangle_{I_j} - \langle f \rangle_{\widehat{I}_j})\sqrt{m(\widehat{I}_j)}\,(h_{I_j} - h_{I_j^b}),$$

where $I^b = \widehat{I} \setminus I \in \mathscr{D}$ is the dyadic brother of $I \in \mathscr{D}$. Using (3.1), (2.3), the assumption that μ is m-increasing and the fact that $m(\widehat{I}) \leq \mu(I)$ for every $I \in \mathscr{D}$ we conclude as desired

$$S_4 \leq \frac{3}{\lambda}\sum_j \|H_{\mathscr{D}}\beta_j\|_{L^1(\mu)} \lesssim \frac{1}{\lambda}\sum_j \langle |f| \rangle_{I_j} m(\widehat{I}_j) \lesssim \frac{1}{\lambda}\sum_j \int_{I_j} |f|\,d\mu \leq \frac{1}{\lambda}\|f\|_{L^1(\mu)}.$$

This completes the proof of (i). $\qquad\square$

Proof of Theorem 2.5, part (ii). Take $f = h_I$ so that $H_{\mathscr{D}}^* f = \sigma(I)\,h_{\widehat{I}}$. Assuming that $H_{\mathscr{D}}^*$ is of weak-type $(1,1)$ we obtain that μ is m-decreasing:

$$2\sqrt{m(\widehat{I})} = \|h_{\widehat{I}}\|_{L^1(\mu)} \approx \|h_{\widehat{I}}\|_{L^{1,\infty}(\mu)} = \|H_{\mathscr{D}}^* f\|_{L^{1,\infty}(\mu)} \lesssim \|h_I\|_{L^1(\mu)} \approx \sqrt{m(I)}.$$

To prove the converse we proceed as above. We shall assume that $\mu \in \mathcal{M}_\infty$, the general case will be considered in Section 3.4 below. The estimates for S_1 and S_2 are standard (since $H_{\mathscr{D}}^*$ is bounded on $L^2(\mu)$). For S_3 we first observe that if $x \in \mathbb{R} \setminus I_j$

$$H_{\mathscr{D}}^* b_j(x) = \sum_{I \in \mathscr{D}} \sigma(I)\langle b_j, h_I\rangle h_{\widehat{I}}(x) = \sigma(I_j)\langle b_j, h_{I_j}\rangle h_{\widehat{I}_j}(x) = \sigma(I_j)\langle f, h_{I_j}\rangle h_{\widehat{I}_j}(x).$$

We use this expression, (2.3) and that μ is m-decreasing:

$$S_3 \leq \frac{3}{\lambda}\sum_j \int_{\mathbb{R}\setminus I_j} |H_{\mathscr{D}}^* b_j(x)|\,d\mu(x) \leq \frac{3}{\lambda}\sum_j \|h_{I_j}\|_{L^\infty(\mu)}\|h_{\widehat{I}_j}\|_{L^1(\mu)}\int_{I_j} |f|\,d\mu$$

$$\approx \frac{1}{\lambda}\sum_j \sqrt{\frac{m(\widehat{I}_j)}{m(I_j)}}\int_{I_j} |f|\,d\mu \lesssim \frac{1}{\lambda}\sum_j \int_{I_j} |f|\,d\mu \leq \frac{1}{\lambda}\|f\|_{L^1(\mu)}.$$

To estimate S_4 we use (3.4),

$$H_{\mathscr{D}}^*\beta_j = \sum_{I \in \mathscr{D}} \sigma(I)\langle \beta_j, h_I\rangle h_{\widehat{I}} = \sigma(\widehat{I}_j)\sigma(I_j)(\langle f \rangle_{I_j} - \langle f \rangle_{\widehat{I}_j})\sqrt{m(\widehat{I}_j)}\,h_{I_j^{(2)}}.$$

We use that μ is m-decreasing and $m(\widehat{I}) \leq \mu(I)$ to conclude that

$$S_4 \leq \frac{3}{\lambda}\sum_j \|H_{\mathscr{D}}^*\beta_j\|_{L^1(\mu)} \leq \frac{12}{\lambda}\sum_j \langle |f| \rangle_{I_j}\sqrt{m(\widehat{I}_j)m(I_j^{(2)})}$$

$$\lesssim \frac{1}{\lambda}\sum_j \langle|f|\rangle_{I_j} m(\widehat{I_j}) \lesssim \frac{1}{\lambda}\sum_j \int_{I_j}|f|\,d\mu \le \frac{1}{\lambda}\|f\|_{L^1(\mu)}.$$

This completes the proof of (ii). $\qquad\square$

Proof of Theorem 2.5, part (iii). If μ is m-increasing we can use (i) to interpolate with the $L^2(\mu)$ bound to conclude estimates on $L^p(\mu)$ for every $1 < p < 2$. Conversely, we note that

$$(3.5) \qquad \|h_I\|_{L^p(\mu)} = \sqrt{m(I)}\left(\frac{1}{\mu(I_-)^{p-1}} + \frac{1}{\mu(I_+)^{p-1}}\right)^{\frac{1}{p}} \approx m(I)^{\frac{1}{2}-\frac{1}{p'}}.$$

On the other hand, if we then assume that $H_{\mathscr{D}}$ is bounded on $L^p(\mu)$ we conclude that

$$m(I_-)^{\frac{1}{2}-\frac{1}{p'}} + m(I_+)^{\frac{1}{2}-\frac{1}{p'}} \approx \|h_{I_-} - h_{I_+}\|_{L^p(\mu)} = \|H_{\mathscr{D}}h_I\|_{L^p(\mu)}$$
$$\lesssim \|h_I\|_{L^p(\mu)} \approx m(I)^{\frac{1}{2}-\frac{1}{p'}}.$$

This and the fact that $1 < p < 2$ imply that μ is m-increasing. $\qquad\square$

Proof of Theorem 2.5, part (iv). For $H_{\mathscr{D}}^*$ we can proceed in the same way. By interpolation and (ii), μ being m-decreasing gives boundedness on $L^p(\mu)$ for $1 < p < 2$. Conversely, if $H_{\mathscr{D}}^*$ is bounded on $L^p(\mu)$ for some $1 < p < 2$, then

$$m(\widehat{I})^{\frac{1}{2}-\frac{1}{p'}} \approx \|h_{\widehat{I}}\|_{L^p(\mu)} = \|H_{\mathscr{D}}^* h_I\|_{L^p(\mu)} \lesssim \|h_I\|_{L^p(\mu)} \approx m(I)^{\frac{1}{2}-\frac{1}{p'}},$$

and therefore μ is m-decreasing. $\qquad\square$

3.3. Haar shift operators in higher dimensions.
We first see that $\text{Ш}_{r,s}$ is a bounded operator on $L^2(\mu)$. Following [12], we write

$$\text{Ш}_{r,s}f(x) = \sum_{Q\in\mathscr{D}}\left(\sum_{\substack{R\in\mathscr{D}_r(Q)\\S\in\mathscr{D}_s(Q)}}\alpha_{R,S}^Q\langle f,\phi_R\rangle\psi_S(x)\right) =: \sum_{Q\in\mathscr{D}}A_Qf(x)$$

As observed before, Φ and Ψ are orthogonal systems. This implies

$$(3.6) \qquad \|A_Qf\|_{L^2(\mu)}^2 = \sum_{S\in\mathscr{D}_s(Q)}\left|\left\langle f, \sum_{R\in\mathscr{D}_r(Q)}\alpha_{R,S}^Q\phi_R\right\rangle\right|^2 \|\psi_S\|_{L^2(\mu)}^2$$
$$\le \|f\|_{L^2(\mu)}^2 \sum_{\substack{R\in\mathscr{D}_r(Q)\\S\in\mathscr{D}_s(Q)}}|\alpha_{R,S}^Q|^2\|\phi_R\|_{L^2(\mu)}^2$$
$$\le 2^{(r+s)d}\Big(\sup_{Q,R,S}|\alpha_{R,S}^Q|^2\Big)\|f\|_{L^2(\mu)}^2.$$

For $Q \in \mathscr{D}$ and non-negative integer r, s, we write $P_{\Phi,Q}^r$ and $P_{\Psi,Q}^s$ for the projections

$$P_{\Phi,Q}^r f = \sum_{R\in\mathscr{D}_r(Q)}\langle f,\phi_R\rangle\phi_R, \qquad P_{\Psi,Q}^s f = \sum_{S\in\mathscr{D}_s(Q)}\langle f,\psi_S\rangle\psi_S.$$

We then have

$$P_{Q,\Psi}^s A_Q P_{Q,\Phi}^r f = \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \langle A_Q(\phi_R), \psi_S\rangle \langle f, \phi_R\rangle \psi_S$$

$$= \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \|\phi_R\|_{L^2(\mu)}^2 \|\psi_S\|_{L^2(\mu)}^2 \alpha_{R,S}^Q \langle f, \phi_R\rangle \psi_S = A_Q f.$$

Fixed r and s, we notice that the projections $P_{\Phi,Q}^r$ are orthogonal on the index Q and the same occurs with $P_{\Psi,Q}^s$. Hence, by (3.6) and orthogonality

$$\|\mathrm{III}_{r,s}f\|_{L^2(\mu)}^2 = \sum_{Q\in\mathscr{D}} \|P_{Q,\Psi}^s A_Q P_{Q,\Phi}^r f\|_{L^2(\mu)}^2 \le C \sum_{Q\in\mathscr{D}} \|P_{Q,\Phi}^r f\|_{L^2(\mu)}^2$$

$$\le C \sum_{Q\in\mathscr{D}} \sum_{R\in\mathscr{D}_r(Q)} |\langle f, \phi_R\rangle|^2 \le C \|f\|_{L^2(\mu)}^2,$$

and this shows that $\mathrm{III}_{r,s}$ is bounded on $L^2(\mu)$.

Proof of Theorem 2.11. We first show that $\Xi(\Phi, \Psi; r, s) < \infty$ implies that $\mathrm{III}_{r,s}$ is of weak-type $(1,1)$. We shall assume that $\mu \in \mathcal{M}_\infty$ and the general case will be considered in Section 3.4 below. Let $\lambda > 0$ be fixed and perform the Calderón-Zygmund decomposition in Theorem 2.1. Then,

$$\mu\{x\in\mathbb{R}^d : |\mathrm{III}_{r,s}f(x)| > \lambda\} \le \mu\{x\in\mathbb{R}^d : |\mathrm{III}_{r,s}g(x)| > \lambda/3\} + \mu(\Omega_\lambda)$$

$$+ \mu\{x\in\mathbb{R}^d\setminus\Omega_\lambda : |\mathrm{III}_{r,s}b(x)| > \lambda/3\}$$

$$+ \mu\{x\in\mathbb{R}^d : |\mathrm{III}_{r,s}\beta(x)| > \lambda/3\}$$

$$= S_1 + S_2 + S_3 + S_4.$$

Using the weak-type $(1,1)$ for $M_\mathscr{D}$, Theorem 2.1 part (a) and that $\mathrm{III}_{r,s}$ is bounded on $L^2(\mu)$ it is standard to check that

$$S_1 + S_2 \le \frac{C_{r,s}}{\lambda}\|f\|_{L^1(\mu)}.$$

We next consider S_3. Let $x\in\mathbb{R}^d\setminus Q_j$ and observe that

$$(3.7)\quad |\mathrm{III}_{r,s}b_j(x)| \le \sup_{Q,R,S} |\alpha_{R,S}^Q| \sum_{\substack{Q\in\mathscr{D}\ R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} |\langle b_j, \phi_R\rangle|\,|\psi_S(x)|$$

$$\lesssim \sum_{Q_j\subsetneq Q\subset Q_j^{(r)}} \sum_{\substack{R\in\mathscr{D}_r(Q), R\subset Q_j\\ S\in\mathscr{D}_s(Q)}} |\langle b_j, \phi_R\rangle|\,|\psi_S(x)|.$$

In the last inequality we have used that each non-vanishing term leads to $Q_j \subsetneq Q \subset Q_j^{(r)}$ and $R\subset Q_j$ since ϕ_R is supported in R and constant on the children of R, b_j is supported in Q_j and has vanishing integral, and ψ_S is supported in S. This, Chebyshev's inequality and Theorem 2.1 imply

$$S_3 \le \frac{3}{\lambda} \sum_j \int_{\mathbb{R}^d\setminus Q_j} |\mathrm{III}_{r,s}b_j|\,d\mu$$

$$\lesssim \frac{1}{\lambda} \sum_j \sum_{Q_j\subsetneq Q\subset Q_j^{(r)}} \sum_{\substack{R\in\mathscr{D}_r(Q), R\subset Q_j\\ S\in\mathscr{D}_s(Q)}} \|b_j\|_{L^1(\mu)}\,\|\phi_R\|_{L^\infty(\mu)}\,\|\psi_S\|_{L^1(\mu)}$$

$$\leq \frac{2^{(r+s)\,d}\,r}{\lambda}\,\Xi(\Phi,\Psi;r,s)\sum_j \|b_j\|_{L^1(\mu)} \leq \frac{C_{r,s}}{\lambda}\,\|f\|_{L^1(\mu)}.$$

We finally estimate S_4. Let us observe that β_j and ϕ_R have vanishing integral. Besides, β_j is supported in $\widehat{Q}_j$ and constant on each dyadic child of $\widehat{Q}_j$, and ϕ_R is supported in R and constant on each dyadic child of R. All these imply that $\langle \beta_j, \phi_R \rangle = 0$ unless $R = \widehat{Q}_j$. Then,

$$(3.8)\quad |\mathrm{III}_{r,s}\beta_j(x)| \leq \sup_{Q,R,S} |\alpha_{R,S}^Q| \sum_{S\in\mathscr{D}_s(Q_j^{(r+1)})} |\langle \beta_j, \phi_{\widehat{Q}_j} \rangle|\,|\psi_S(x)|$$

$$\lesssim \|\beta_j\|_{L^1(\mu)} \sum_{S\in\mathscr{D}_s(Q_j^{(r+1)})} \|\phi_{\widehat{Q}_j}\|_{L^\infty(\mu)}\,|\psi_S(x)|.$$

Therefore, Chebyshev's inequality and Theorem 2.1 imply

$$S_4 \leq \frac{3}{\lambda}\sum_j \|\mathrm{III}_{r,s}\beta_j\|_{L^1(\mu)}$$

$$\lesssim \frac{1}{\lambda}\sum_j \|\beta_j\|_{L^1(\mu)} \sum_{S\in\mathscr{D}_s(Q_j^{(r+1)})} \|\phi_{\widehat{Q}_j}\|_{L^\infty(\mu)} \|\psi_S\|_{L^1(\mu)}$$

$$\leq \frac{2^{sd}}{\lambda}\Xi(\Phi,\Psi;r,s)\sum_j \|\beta_j\|_{L^1(\mu)} \leq \frac{C_{r,s}}{\lambda}\,\|f\|_{L^1(\mu)}.$$

Gathering the obtained estimates this part of the proof is complete.

We now turn to the converse, that is, we show that if a non-degenerate Haar shift $\mathrm{III}_{r,s}$ is of weak-type $(1,1)$ then $\Xi(\Phi,\Psi;r,s) < \infty$. For every $Q \in \mathscr{D}_\Phi$, we pick $Q_\infty \in \mathscr{D}_1(Q)$ such that ϕ_Q (which we recall that is constant in the dyadic children of Q) attains its maximum in Q_∞. Define

$$\widetilde{\varphi}_Q(x) = (\varphi_Q(x) - \langle \varphi_Q \rangle_Q)\,1_Q(x), \quad \varphi_Q(x) = \mathrm{sgn}\,(\phi_Q(x))\frac{1_{Q_\infty}(x)}{\mu(Q_\infty)},$$

where $\mathrm{sgn}(t) = t/|t|$ if $t \neq 0$ and $\mathrm{sgn}(0) = 0$. We note that by construction $\widetilde{\varphi}_Q$ is supported on Q, constant on dyadic children of Q and has vanishing integral. These imply that $\langle \widetilde{\varphi}_Q, \phi_R \rangle = 0$ if $Q \neq R$. Also,

$$\langle \widetilde{\varphi}_Q, \phi_Q \rangle = \langle \varphi_Q, \phi_Q \rangle = \frac{1}{\mu(Q_\infty)} \int_{Q_\infty} |\phi_Q(x)|\,d\mu(x) = \|\phi_Q\|_{L^\infty(\mu)},$$

where we have used that ϕ_Q has vanishing integral and is constant on the dyadic children of Q. On the other hand,

$$\|\widetilde{\varphi}_Q\|_{L^1(\mu)} \leq 2 \int_Q |\varphi_Q(x)|\,d\mu(x) = 2.$$

Let us now obtain that $\Xi(\Phi,\Psi;r,s) < \infty$. In the definition of $\Xi(\Phi,\Psi;r,s)$ we may clearly assume that $R \in \mathscr{D}_\Phi$ and $S \in \mathscr{D}_\Psi$. Thus, we fix $Q_0 \in \mathscr{D}$, $R_0 \in \mathscr{D}_r(Q_0)$ and $S_0 \in \mathscr{D}_s(Q_0)$ with $\|\phi_{R_0}\|_{L^2(\mu)} = 1$ and $\|\psi_{S_0}\|_{L^2(\mu)} = 1$. We use the properties of the function $\widetilde{\varphi}_{R_0}$ just defined and the non-degeneracy of $\mathrm{III}_{r,s}$ to obtain that for every $x \in \mathbb{R}^d$

$$|\text{III}_{r,s}\widetilde{\varphi}_{R_0}(x)| = \Big| \sum_{Q\in\mathscr{D}} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha_{R,S}^{Q} \langle\widetilde{\varphi}_{R_0}, \phi_R\rangle \psi_S(x) \Big|$$

$$= \|\phi_{R_0}\|_{L^\infty(\mu)} \Big| \sum_{S\in\mathscr{D}_s(Q_0)} \alpha_{R_0,S}^{Q_0}\psi_S(x) \Big| \geq \inf_{Q,R,S}|\alpha_{R,S}^{Q}| \, \|\phi_{R_0}\|_{L^\infty(\mu)} \, |\psi_{S_0}(x)|.$$

Using that $\text{III}_{r,s}$ is of weak-type $(1,1)$ and that ψ_{S_0} is constant on dyadic children of S_0 we obtain

$$\|\phi_{R_0}\|_{L^\infty(\mu)} \, \|\psi_{S_0}\|_{L^1(\mu)} \approx \big\| \|\phi_{R_0}\|_{L^\infty(\mu)} \, \psi_{S_0}\big\|_{L^{1,\infty}(\mu)}$$
$$\lesssim \|\text{III}_{r,s}\widetilde{\varphi}_{R_0}\|_{L^{1,\infty}(\mu)} \lesssim \|\widetilde{\varphi}_{R_0}\|_{L^1(\mu)} \leq 2.$$

This immediately implies that $\Xi(\Phi,\Psi;r,s) < \infty$. $\qquad\qquad\square$

Remark 3.9. From the previous proof and a standard homogeneity argument on the parameter $\|\text{III}_{r,s}\|_{L^2(\mu)\to L^2(\mu)}$ we obtain that, under the conditions of Theorem 2.11,

$$\|\text{III}_{r,s}\|_{L^1(\mu)\to L^{1,\infty}(\mu)} \leq C_0 \big(\|\text{III}_{r,s}\|_{L^2(\mu)\to L^2(\mu)}$$
$$+ 2^{s\,d}\,(r\,2^{r\,d}+1)\,\Xi(\Phi,\Psi;r,s) \sup_{Q,R,S}|\alpha_{R,S}^{Q}|\big),$$

where C_0 is a universal constant (independent of the dimension, for instance, in the previous argument one can safely take $C_0 \leq 217$.)

Remark 3.10. One can obtain an analog of Theorem 2.5 parts (iii), (iv) for non-degenerate Haar shift operators defined in terms of 2-value Haar systems Φ and Ψ. To be more precise, let $\text{III}_{r,s}$ be a non-degenerate Haar shift of complexity (r,s) associated to two 2-value generalized Haar systems. If $\text{III}_{r,s}$ is of weak-type (p,p) for some $1 < p < 2$ then $\Xi(\Phi,\Psi;r,s) < \infty$. The proof is very similar to what we did for the dyadic Hilbert transform. Fix $Q_0 \in \mathscr{D}$, $R_0 \in \mathscr{D}_r(Q_0)$, $S_0 \in \mathscr{D}_s(Q_0)$. Then

$$|\text{III}_{r,s}\phi_{R_0}(x)| = \Big| \sum_{Q\in\mathscr{D}} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha_{R,S}^{Q} \delta_{R_0,R}\,\psi_S(x) \Big|$$

$$= \Big| \sum_{S\in\mathscr{D}_s(Q_0)} \alpha_{R_0,S}^{Q_0}\psi_S(x) \Big| \geq \inf_{Q,R,S}|\alpha_{R,S}^{Q}|\,|\psi_{S_0}(x)|.$$

Using that $\text{III}_{r,s}$ is of weak-type (p,p) and that ψ_{S_0} is constant on dyadic children of S_0 we obtain

$$\|\psi_{S_0}\|_{L^p(\mu)} \lesssim \|\text{III}_{r,s}\phi_{R_0}\|_{L^{p,\infty}(\mu)} \lesssim \|\phi_{R_0}\|_{L^p(\mu)}.$$

Also, by (2.9), (2.10) and proceeding as in (3.5) we obtain

$$\|\psi_{S_0}\|_{L^1(\mu)}^{1-\frac{2}{p'}} \approx m_\Psi(S_0)^{\frac{1}{2}-\frac{1}{p'}} \approx \|\psi_{S_0}\|_{L^p(\mu)}$$
$$\lesssim \|\phi_{R_0}\|_{L^p(\mu)} \approx m_\Phi(R_0)^{\frac{1}{2}-\frac{1}{p'}} \approx \|\phi_{R_0}\|_{L^\infty(\mu)}^{-(1-\frac{2}{p'})}.$$

This easily implies that $\Xi(\Phi,\Psi;r,s) < \infty$.

3.4. **The case** $\mu \in \mathcal{M} \setminus \mathcal{M}_\infty$. The Calderón-Zygmund decomposition in Theorem 2.1 has been obtained under the assumption that every d-dimensional quadrant has infinite μ-measure, $\mu \in \mathcal{M}_\infty$ in the language of Section 3.1. Also, Theorems 2.5 and 2.11 have been proved under this assumption. Here we discuss how to remove this constraint and work with arbitrary measures in $\mathcal{M}$.

Due to the nature of the standard dyadic grid, $\mathbb{R}^d$ splits naturally in 2^d components each of them being a d-dimensional quadrant. Let $\mathbb{R}_k^d$, $1 \leq k \leq 2^d$, denote the d-dimensional quadrants in $\mathbb{R}^d$: that is, the sets $\mathbb{R}^\pm \times \cdots \times \mathbb{R}^\pm$ where $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^- = (-\infty, 0)$. Let $\mathscr{D}^k$ be the collection of dyadic cubes contained in $\mathbb{R}_k^d$. We set

$$M_{\mathscr{D}^k} f(x) = \sup_{x \in Q \in \mathscr{D}^k} \frac{1}{|Q|} \int_Q |f(y)| \, d\mu(y) = M_{\mathscr{D}}(f \, 1_{\mathbb{R}_k^d})(x) \, 1_{\mathbb{R}_k^d}(x).$$

Hence, given a function f we have that

$$f(x) = \sum_{k=1}^{2^d} f(x) \, 1_{\mathbb{R}_k^d}(x), \qquad M_{\mathscr{D}} f(x) = \sum_{k=1}^{2^d} M_{\mathscr{D}^k} f(x) \, 1_{\mathbb{R}_k^d}(x),$$

and in each sum there is at most only one non-zero term. Because of this decomposition, to extend our results it will suffice to assume that f is supported in some $\mathbb{R}_k^d$ and obtain the corresponding decompositions and estimates in $\mathbb{R}_k^d$.

Notice that if f is supported in $\mathbb{R}_k^d$, $M_{\mathscr{D}} f = M_{\mathscr{D}^k} f$ and this function is supported in $\mathbb{R}_k^d$. In particular, for any $\lambda > 0$,

$$\Omega_\lambda = \{x \in \mathbb{R}^d : M_{\mathscr{D}} f(x) > \lambda\} = \{x \in \mathbb{R}_k^d : M_{\mathscr{D}^k} f(x) > \lambda\},$$

and so any decomposition of this set will consist of cubes in $\mathscr{D}^k$. We modify our notation and define $\langle f \rangle_{\mathbb{R}_k^d} = \frac{1}{\mu(\mathbb{R}_k^d)} \int_{\mathbb{R}_k^d} f \, d\mu$ if $\mu(\mathbb{R}_k^d) < \infty$ and $\langle f \rangle_{\mathbb{R}_k^d} = 0$ if $\mu(\mathbb{R}_k^d) = \infty$.

The following result is the analog of Theorem 2.1.

Theorem 3.11. *Given $1 \leq k \leq 2^d$, $\mu \in \mathcal{M}$ and $f \in L^1(\mu)$ with $\operatorname{supp} f \subset \mathbb{R}_k^d$, so that for every $\lambda > \langle |f| \rangle_{\mathbb{R}_k^d}$ there exists a covering of $\Omega_\lambda = \{M_{\mathscr{D}} f > \lambda\}$ by maximal dyadic cubes $\{Q_j\}_j \subset \mathscr{D}^k$. Then, we may find a decomposition $f = g + b + \beta$ with g, b and β as defined in Theorem 2.1 and satisfying the very same properties.*

Proof. If $\mu(\mathbb{R}_k^d) = \infty$, then the proof given above goes through without change. If $\mu(\mathbb{R}_k^d) < \infty$, then in the notation used above, $\langle |f| \rangle_Q \to \langle |f| \rangle_{\mathbb{R}_k^d} < \lambda$ as $\ell(Q) \to \infty$ for $Q \in \mathscr{D}^k$. Hence, if $Q \in \mathscr{D}^k$ is such that $\langle |f| \rangle_Q > \lambda$, then Q must be contained in a maximal cube with the same property. Hence, we can easily form the collection of maximal cubes $\{Q_j\}_j \subset \mathscr{D}^k$. We observe that this covering gives the right estimate for the level sets of $M_{\mathscr{D}} f = M_{\mathscr{D}^k} f$ if $\lambda > \langle |f| \rangle_{\mathbb{R}_k^d}$. For $0 < \lambda \leq \langle |f| \rangle_{\mathbb{R}_k^d}$ we immediately have

$$\mu(\Omega_\lambda) \leq \mu(\mathbb{R}_k^d) \leq \frac{1}{\lambda} \int_{\mathbb{R}_k^d} |f(x)| \, d\mu(x).$$

These in turn imply that $M_{\mathscr{D}^j}$ is of weak-type $(1,1)$. From here we repeat the arguments in the proof Theorem 2.1 to complete the proof without change. □

Proof of Theorems 2.5 and 2.11 for $\mu \in \mathcal{M}$. We obtain the weak-type $(1,1)$ estimate for $\mathrm{III}_{r,s}$, the arguments for $H_{\mathscr{D}}$ and $H_{\mathscr{D}}^*$ are identical.

Suppose first that $\operatorname{supp} f \subset \mathbb{R}_k^d$ with $1 \le k \le 2^d$. If $\mu(\mathbb{R}_k^d) = \infty$, then the arguments above go through without change. Assume otherwise that $\mu(\mathbb{R}_k^d) < \infty$. If $\lambda > \langle |f| \rangle_{\mathbb{R}_k^d}$ then we repeat the same proof using Theorem 3.11 in place of Theorem 2.1. If $0 < \lambda \le \langle |f| \rangle_{\mathbb{R}_k^d}$ we cannot form the Calderón-Zygmund decomposition. Nevertheless, the estimate is immediate after observing that by construction $\mathrm{III}_{r,s}f$ is supported in $\mathbb{R}_k^d$ since so is f. Then,

$$\mu(\{x \in \mathbb{R}^d : |\mathrm{III}_{r,s}f(x)| > \lambda\}) \le \mu(\mathbb{R}_k^d) \le \frac{1}{\lambda} \int_{\mathbb{R}_k^d} |f(x)| \, d\mu(x).$$

To prove the weak-type estimate in the general case, fix f and write $f = \sum_{k=1}^{2^d} f \, 1_{\mathbb{R}_k^d}$. By construction we then have

$$\mathrm{III}_{r,s}f(x) = \sum_{k=1}^{2^d} \mathrm{III}_{r,s}(f \, 1_{\mathbb{R}_k^d})(x) \, 1_{\mathbb{R}_k^d}(x).$$

Therefore, by the above argument applied to each $\mathbb{R}_k^d$, we conclude as desired

$$\mu(\{x \in \mathbb{R}^d : |\mathrm{III}_{r,s}f(x)| > \lambda\}) = \sum_{k=1}^{2^d} \mu(\{x \in \mathbb{R}_k^d : |\mathrm{III}_{r,s}(f \, 1_{\mathbb{R}_k^d})(x)| > \lambda\})$$

$$\lesssim \frac{1}{\lambda} \sum_{k=1}^{2^d} \int_{\mathbb{R}_k^d} |f(x)| \, d\mu(x) = \frac{1}{\lambda} \int_{\mathbb{R}^d} |f(x)| \, d\mu(x).$$

□

Remark 3.12. As explained above, the standard dyadic grid splits $\mathbb{R}^d$ in 2^d components, each of them being a d-dimensional quadrant. These components are defined with respect to the property that if a given cube is in a fixed component, all of its relatives (ascendants and descendants) remain in the same component. This connectivity property depends on the dyadic grid chosen, and one can find other dyadic grids with other number of components. Let us work for simplicity in $\mathbb{R}$ and suppose that we want to find dyadic grids "generated" by $I_0 = [0, 1)$. We need to give the ascendants of I_0, say I_k, $k \le -1$. Once we have them, we translate each I_k by $j \, 2^{-k}$ with $j \in \mathbb{Z}$ and these define the cubes of the fixed generation 2^{-k} for $k \le 0$. The small cubes are obtained by subdivision. Hence, in the present scenario, we only need to define the I_k's. Let us start by finding the parent of I_0: we just have two choices $[0, 2)$ or $[-1, 1)$, and once we choose one, which we call I_{-1}, we need to pass to the next level and decide which is the parent of I_{-1}, for which again we have to choices. Continuing this we have a sequence of cubes I_k, $k \le 0$, which determines the dyadic grid. In the classical dyadic grid one always choose the parent of I_k "to the right", that is, so that I_k is the left half

of I_{k-1}. This eventually gives two components. One way to obtain a dyadic grid with one component is to alternatively take parents "to the left" and "to the right". That is, if we take $I_0 = [0, 1)$, $I_{-1} = [-1, 1)$, $I_{-2} = [-1, 3)$, $I_{-3} = [-5, 3)$, ... we obtain one component. More precisely, take the family of intervals $I_k = [0, 2^{-k})$ for $k \geq 0$ and for $k \leq -1$ let $I_k = [a_k - 2^{-k}, a_k)$ with $a_k = (2^{-k} + 1)/3$ if $-k$ is odd and $a_k = (2^{-k+1} + 1)/3$ if $-k$ is even. Notice that $\{I_k\}_{k \in \mathbb{Z}}$ is a decreasing family of intervals of dyadic side-length. Notice that each I_k is one of the halves of I_{k-1}. Using I_k we generate the dyadic cubes of generation 2^{-k} by taking the intervals $I_{j,k} = j\, 2^{-k} + I_k$ with $j \in \mathbb{Z}$. Finally we set $\widetilde{\mathscr{D}} = \{I_{j,k} : j, k \in \mathbb{Z}\}$. This is clearly a dyadic grid in $\mathbb{R}$. Let us observe that $a_k \to \infty$ and $a_k - 2^{-k} \to -\infty$ as $k \to -\infty$ and therefore $I_k \nearrow \mathbb{R}$ as $k \to -\infty$. This means that this dyadic grid induces just one component (in the sense described above) since for any $I_1, I_2 \in \widetilde{\mathscr{D}}$ we can find a large k such that both I_1 and I_2 are contained in $I_{-k} \in \widetilde{\mathscr{D}}$. We finally observe that the dyadic grids with one component occur more often than those with two, as the classical dyadic grid. Indeed, if at each generation we select randomly the parent (among the possibilities "to the left" and "to the right"), the probability of ending with a system with one component is 0.

4. Examples of measures and Haar systems

4.1. **The 1-dimensional case.** As we have seen above the 1-dimensional case is somehow special since the Haar system is "uniquely" determined. Let us work with the measures in Theorem 2.5, that is, μ is a Borel measure in $\mathbb{R}$ with $0 < \mu(I) < \infty$ for every $I \in \mathscr{D}$. As we have seen in that result, m-increasing, m-decreasing and m-equilibrated measures are the ones governing the boundedness of $H_{\mathscr{D}}$, $H_{\mathscr{D}}^*$ and Haar shift operators. We are going to describe some examples of non-standard measures satisfying those conditions.

We can easily obtain examples of m-equilibrated measures. Let μ be a dyadically doubling measure, i.e., $\mu(\widehat{I}) \lesssim \mu(I)$ for all $I \in \mathscr{D}$ where $\widehat{I}$ is the dyadic parent of I. Then, $m(I) \approx \mu(I)$ and clearly μ is m-equilibrated. This applies straightforwardly to the Lebesgue measure.

We next construct some measures that are m-increasing, m-decreasing or m-equilibrated without being dyadically doubling or of polynomial growth. Set $d\nu = dx\,1_{\mathbb{R}\setminus[0,1)} + d\mu$, where μ is a measure supported on the interval $[0, 1)$ defined as follows. Let $\{I_k\}_{k \geq 0}$ be the decreasing sequence of dyadic intervals $I_k = [0, 2^{-k})$ and let $\{a_k\}_{k \geq 1}$ be such that $0 < a_k < 1$ and $a_1 = 1/2$. Set $b_k = 1 - a_k$. Define μ recursively by setting $\mu(I_0) = 1$ and

$$(4.1) \quad \mu(I_k) = a_k \mu(\widehat{I}_k) = a_k \mu(I_{k-1}) \quad \text{and} \quad \mu(I_k^b) = b_k \mu(\widehat{I}_k) = b_k \mu(I_{k-1}),$$

for $k \geq 1$, where we recall that $I_k^b = [2^{-k}, 2^{-k+1})$ is the dyadic brother of I_k. On I_k^b, μ is taken to be uniform, i.e., $\mu(J) = \mu(I_k^b)\,|J|/|I_k^b|$ for any $J \in \mathscr{D}$, $J \subset I_k^b$. We illustrate this procedure in Figure 1.
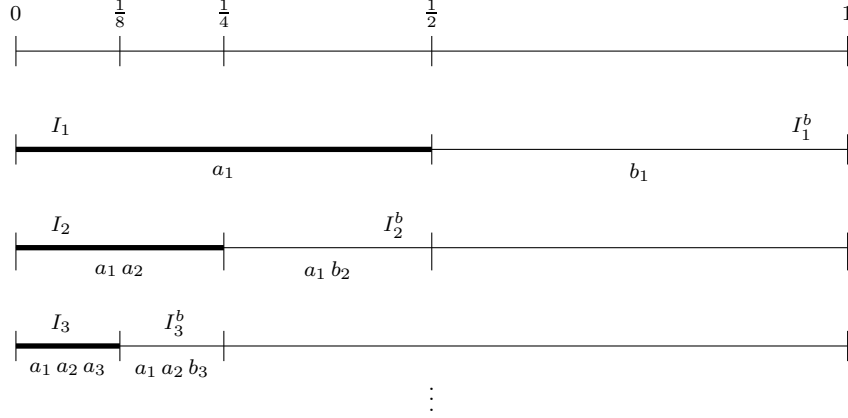
FIGURE 1. Construction of μ

By construction, if $I \cap I_0 = \emptyset$ or $I_0 \subset I$ we have

$$\frac{m(I)}{m(\widehat{I})} = \frac{|I|/4}{|\widehat{I}|/4} = \frac{1}{2}$$

Also, if $I \in \mathscr{D}$ and $\widehat{I} \subset I_k^b$ for some $k \geq 1$ then

$$\frac{m(I)}{m(\widehat{I})} = \frac{\frac{\mu(I_k^b)\,|I|}{4\,|I_k^b|}}{\frac{\mu(I_k^b)\,|\widehat{I}|}{4\,|I_k^b|}} = \frac{1}{2}$$

In the remainder cases we always have that $\widehat{I} = \widehat{I}_k$ for some $k \geq 1$ and I is either I_k or I_k^b. Note that by (4.1) we get

$$m(I_k^b) = \frac{\mu((I_k^b)_-)\mu((I_k^b)_+)}{\mu(I_k^b)} = \frac{\mu(I_k^b)}{4} = \frac{1}{4}\,b_k\,\mu(\widehat{I}_k),$$

$$m(I_k) = \frac{\mu((I_k)_-)\mu((I_k)_+)}{\mu(I_k)} = \frac{\mu(I_{k+1})\mu(I_{k+1}^b)}{\mu(I_k)} = a_{k+1}\,b_{k+1}\,a_k\,\mu(\widehat{I}_k),$$

$$m(\widehat{I}_k) = \frac{\mu(I_k)\mu(I_k^b)}{\mu(\widehat{I}_k)} = a_k\,b_k\,\mu(\widehat{I}_k).$$

Hence,

$$(4.2) \qquad \frac{m(I_k)}{m(\widehat{I}_k)} = \frac{a_{k+1}b_{k+1}}{b_k} \qquad \text{and} \qquad \frac{m(I_k^b)}{m(\widehat{I}_k)} = \frac{1}{4a_k}.$$

We now proceed to study the previous ratios associated to measures given by particular choices of the defining sequences $\{a_k\}_k$ and $\{b_k\}_k$. We shall construct three non-dyadically doubling and of non-polynomial growth measures. In the first example μ is m-equilibrated, in the second μ is m-increasing and is not m-decreasing, in the third μ is m-decreasing and is not m-increasing. Finally, in the last example we give a measure μ which is of

polynomial growth but is neither dyadically doubling, nor m-increasing, nor m-decreasing.

(a) Let $b_k = \frac{1}{k}$ for $k \geq 2$. The measure μ is **non-dyadically doubling** since by (4.1), if $k \geq 2$

$$\frac{\mu(\widehat{I}_k)}{\mu(I_k^b)} = \frac{1}{b_k} = k \xrightarrow[k \to \infty]{} \infty.$$

From substituting a_k and b_k in (4.2) we get that,

$$\frac{m(I_k)}{m(\widehat{I}_k)} = \left(1 - \frac{1}{k+1}\right)\frac{k}{k+1}, \qquad \frac{m(I_k^b)}{m(\widehat{I}_k)} = \frac{1}{4\left(1 - \frac{1}{k}\right)}.$$

Both sequences are bounded from above and from below, which implies that μ is m-**equilibrated**. Besides, for $0 < t < \infty$

$$\frac{\mu(I_k)}{|I_k|^t} = \frac{a_1 \dots a_k}{2^{-kt}} = \frac{1}{2}\frac{2^{kt}}{k} \xrightarrow[k \to \infty]{} \infty.$$

Thus, μ does **not have polynomial growth**.

(b) Set $b_k = 2^{-k^2}$. In this case μ is **non-dyadically doubling**, since by (4.1)

$$\frac{\mu(\widehat{I}_k)}{\mu(I_k^b)} = 2^{k^2} \xrightarrow[k \to \infty]{} \infty.$$

Since $\frac{1}{2} \leq a_k < 1$, by (4.2) we get that $m(\widehat{I}_k) \approx m(I_k^b)$. However,

$$4 < \frac{m(\widehat{I}_k)}{m(I_k)} = \frac{2^{-k^2}}{\left(1 - 2^{-(k+1)^2}\right)2^{-(k+1)^2}} \xrightarrow[k \to \infty]{} \infty.$$

Thus, μ is m-**increasing** but is **not m-decreasing**. Notice that for $t > 1$,

$$\frac{\mu(I_k)}{|I_k|^t} = \frac{a_1 \dots a_k}{2^{-kt}} = 2^{kt}\prod_{j=1}^{k}(1 - 2^{-j^2}) \geq 2^{kt}\left(1 - \frac{1}{2}\right)^k = 2^{k(t-1)} \xrightarrow[k \to \infty]{} \infty.$$

For $0 < t \leq 1$, let n and m be positive integers such that $\frac{1}{n+1} < t \leq \frac{1}{n}$ and $k = 2(n+1)m$. Then, $2^{kt} > 2^{2m}$ and

$$\frac{\mu(I_k)}{|I_k|^t} \geq \left(2^m\prod_{j=1}^{m}(1 - 2^{-j^2})\right) \cdot \left(2^m\prod_{j=m+1}^{k}(1 - 2^{-j^2})\right) \geq 2^m\prod_{j=m+1}^{k}(1 - 2^{-j^2})$$

$$\geq 2^m\left(1 - 2^{-m^2}\right)^{k-m} = \left(2\left(1 - 2^{-m^2}\right)^{(2(n+1)-1)}\right)^m \xrightarrow[m \to \infty]{} \infty$$

Thus, μ does **not have polynomial growth**.

(c) Let $n \in \mathbb{N}$ and set $f(n) = \frac{n(n+1)}{2}$. For $k \geq 2$ define

$$b_k = \frac{1}{2}\frac{1}{k - f(n-1)},$$

where $n \geq 2$ is such that $f(n-1) < k \leq f(n)$. Fix $n \geq 2$ and $f(n-1) < k \leq f(n)$. Then $k = f(n-1) + r$, with $1 \leq r \leq f(n) - f(n-1) = n$

and $b_k = 1/(2r)$. Hence,

$$\frac{1}{2n} \le b_k \le \frac{1}{2}.$$

and $\liminf_{k \to \infty} b_k = 0$. By (4.1) this choice of b_k defines a **non-doubling** measure. Since $\frac{1}{2} \le a_k < 1$, by (4.2) we get that $m(\widehat{I}_k) \approx m(I_k^b)$ for every k. On the other hand,

$$\frac{b_{k+1}}{b_k} = \begin{cases} \dfrac{k - f(n-1)}{k+1-f(n-1)} = \dfrac{r}{r+1} \approx 1, & \text{if } k < f(n); \\[2ex] \dfrac{k-f(n-1)}{k+1-f(n)} = n \to \infty, & \text{if } k = f(n). \end{cases}$$

Hence, by (4.2) μ is **not** m**-increasing**. However, μ is m**-decreasing** since $b_k/b_{k+1} \le 2$.

We finally see that μ has no polynomial growth. We start with the case $t > 1$. For $s, j \ge 2$ such that $f(s-1) < j = f(s-1) + r \le f(s)$ with $1 \le r \le s$, we have that $a_j = \frac{2r-1}{2r}$. Then, if $k = f(n)$

$$\frac{\mu(I_k)}{|I_k|^t} = \frac{a_1 \dots a_k}{2^{-kt}} = 2^{kt} \prod_{s=1}^{n} \prod_{r=1}^{s} \frac{2r-1}{2r} = 2^{k(t-1)} \prod_{s=1}^{n} \prod_{r=1}^{s} \frac{2r-1}{r} \ge 2^{k(t-1)}$$
$$= 2^{f(n)\,(t-1)} \xrightarrow[n\to\infty]{} \infty.$$

Consider now $0 < t \le 1$ and let $m \ge 2$ be the unique integer such that $\frac{2}{f(m)} < t \le \frac{2}{f(m-1)}$. Let $k = f(n)$ with n large enough so that $k \ge f(m)^2$. Then $2^{kt} \ge 2^{2f(m)}$ and

$$\frac{\mu(I_k)}{|I_k|^t} \ge \left(2^{f(m)} \prod_{s=1}^{m} \prod_{r=1}^{s} \frac{2r-1}{2r} \right) \cdot \left(2^{f(m)} \prod_{s=m+1}^{n} \prod_{r=1}^{s} \frac{2r-1}{2r} \right)$$
$$\ge 2^{f(m)} \prod_{s=m+1}^{n} \prod_{r=1}^{s} \frac{2r-1}{2r}$$
$$= 2^{f(m)} \prod_{s=m+1}^{n} \frac{(2s)!}{2^{2s}(s!)^2}$$
$$= 2^{f(m)} 2^{-2(f(n)-f(m))} \prod_{s=m+1}^{n} \frac{(2s)!}{(s!)^2}$$
$$\ge 2^{f(m)} 2^{-2(f(n)-f(m))} 2^{3(f(n)-f(m))} = 2^{f(n)} \xrightarrow[n\to\infty]{} \infty,$$

where in the last inequality we have used that $(2s)!/(s!)^2$ is increasing and therefore bounded from below by 8. Thus, μ does **not have polynomial growth**.

(d) Let $b_2 = b_3 = 1/2$, and for every $k \ge 2$, $b_{2k} = 1/k$, $b_{2k+1} = 1 - 1/k$. The measure μ is **non-dyadically doubling** since by (4.1), if $k \ge 2$, then

$$\frac{\mu(\widehat{I}_{2k})}{\mu(I_{2k}^b)} = \frac{1}{b_{2k}} = k \xrightarrow[k\to\infty]{} \infty.$$

From substituting a_k and b_k in (4.2) we get that,

$$\frac{m(I_{2k+1}^b)}{m(\widehat{I}_{2k+1})} = \frac{1}{4a_{2k+1}} = \frac{k}{4} \xrightarrow[k \to \infty]{} \infty,$$

which implies that μ is **not m-increasing**. Also,

$$\frac{m(\widehat{I}_{2k+1})}{m(I_{2k+1})} = \frac{b_{2k+1}}{a_{2(k+1)}\, b_{2(k+1)}} = \frac{(k+1)^2\,(k-1)}{k^2} \xrightarrow[k \to \infty]{} \infty,$$

which implies that μ is **not m-decreasing**.

We finally see that μ **has linear growth**, that is, $\mu(I)/|I| \leq C$ for every I. We first notice that it suffices to consider $I \in \mathscr{D}$ since any arbitrary interval J can be covered by a bounded number of $I \in \mathscr{D}$ with $|I| \approx |J|$. Let us now fix $I \in \mathscr{D}$. The cases $I \cap [0,1) = \emptyset$ or $[0,1) \subset I$ are trivial since $\mu(I) = |I|$. Suppose next that $I \subsetneq [0,1)$. Then, either $I = I_k$ or $I \subset I_k^b$ for some $k \geq 1$. In the latter scenario we have that by construction $\mu(I)/|I| = \mu(I_k^b)/|I_k^b|$, therefore we only have to consider $I = I_k$ or $I = I_k^b$ for k large. Let us fix $k \geq 6$. Notice that

$$\frac{\mu(I_k^b)}{|I_k^b|} = \frac{\mu(I_k)}{|I_k|}\,\frac{b_k}{a_k} = \frac{\mu(I_{k-1})}{|I_{k-1}|}\, 2\, b_k.$$

Thus,

$$\frac{\mu(I_{2k}^b)}{|I_{2k}^b|} = \frac{\mu(I_{2k})}{|I_{2k}|}\,\frac{b_{2k}}{a_{2k}} \leq \frac{\mu(I_{2k})}{|I_{2k}|}, \qquad \frac{\mu(I_{2k+1}^b)}{|I_{2k+1}^b|} = \frac{\mu(I_{2k})}{|I_{2k}|}\, 2\, b_{2k+1} \leq 2\frac{\mu(I_{2k})}{|I_{2k}|}.$$

Additionally,

$$\frac{\mu(I_{2k+1})}{|I_{2k+1}|} = \frac{\mu(I_{2k})}{|I_{2k}|}\, 2\, a_{2k+1} \leq 2\,\frac{\mu(I_{2k})}{|I_{2k}|}.$$

All these together show that it suffices to bound $\mu(I_{2k})/|I_{2k}|$ for $k \geq 3$. Let $k \geq 3$, then we obtain as desired

$$\mu(I_{2k}) = \prod_{j=1}^{2k} a_j = 2^{-3}\Big(\prod_{j=2}^{k} a_{2j}\Big)\Big(\prod_{j=2}^{k-1} a_{2j+1}\Big) = 2^{-3}\frac{1}{k!} \leq \frac{4}{3}\, 2^{-2k} = \frac{4}{3}\,|I_{2k}|.$$

4.2. The higher dimensional case: specific Haar system constructions.

As we have shown in Theorem 2.11, the weak-type $(1,1)$ estimate for Haar shifts is governed by the finiteness of the quantities $\Xi(\Phi, \Psi; r, s)$. In the 1-dimensional case, these can be written only in terms of the measure μ since the Haar system $\mathcal{H}$ is "unique" (see Remark 2.8). However in higher dimensions we have different choices of the Haar system and each of them may lead to a different condition. Therefore, before getting into that let us construct some specific Haar systems.

Among the μ-Haar systems in higher dimensions, two of them are relatively easy to construct: Wilson's Haar system and Mitrea's Haar system [23], [7], [3], [16], [14]. Following [9], we present a simplified way of obtaining this two μ-Haar systems for measures $\mu \in \mathcal{M}$.

To construct Wilson's Haar system, start with some enumeration $(Q_j)_{j=1}^{2^d}$ of the dyadic children of Q and build a dyadic (or logarithmic) partition tree on it. The partition is given as follows: set $\mathscr{W}_0(Q) = \{\{1, 2, \ldots, 2^d\}\}$ and

let $\mathscr{W}_1(Q) = \{\{1, \ldots, 2^{d-1}\}, \{2^{d-1} + 1, \ldots, 2^d\}\}$. Proceed recursively to get the partition $\mathscr{W}_k(Q)$, obtained upon halving the elements of $\mathscr{W}_{k-1}(Q)$ and ending up with $\mathscr{W}_d(Q) = \{\{1\}, \{2\}, \ldots, \{2^d\}\}$. Set

$$E_Q^\omega = \bigcup_{j \in \omega} Q_j \qquad \text{with} \qquad \omega \in \mathscr{W}(Q) = \bigcup_{k=0}^{d-1} \mathscr{W}_k(Q).$$

We are going to see that the family of sets $\{E_Q^\omega\}_{\omega \in \mathscr{W}(Q)}$ behaves like a 1-dimensional dyadic grid. Form construction, any $\omega \in \mathscr{W}_{k-1}(Q)$, $1 \leq k \leq d$, has two disjoint children $\omega_-, \omega_+ \in \mathscr{W}_k(Q)$ such that $\omega = \omega_- \cup \omega_+$. Thus, following the notation of the 1-dimensional case, we write $(E_Q^\omega)_- = E_Q^{\omega_-}$ and $(E_Q^\omega)_+ = E_Q^{\omega_+}$. Note that these two sets are disjoint and $E_Q^\omega = (E_Q^\omega)_- \cup (E_Q^\omega)_+$. We call $(E_Q^\omega)_-$ and $(E_Q^\omega)_+$ the dyadic children of E_Q^ω. Besides, for every $\omega \in \mathscr{W}_k(Q)$, $1 \leq k \leq d$, there exists a unique $\widehat{\omega} \in \mathscr{W}_{k-1}(Q)$ such that $\widehat{\omega} \supset \omega$ and thus $E_Q^\omega \subset \widehat{E_Q^\omega} = E_Q^{\widehat{\omega}}$. We call $\widehat{E_Q^\omega}$ the dyadic parent of E_Q^ω. Moreover, E_Q^ω and $E_Q^{\omega'}$ are either disjoint or one is contained in the other.

We define the Haar functions adapted to the family of sets $\{E_Q^\omega\}_{\omega \in \mathscr{W}(Q)}$: for every $\omega \in \mathscr{W}(Q)$ we set

$$h_Q^\omega = \sqrt{m(E_Q^w)}\left(\frac{1_{(E_Q^\omega)_-}}{\mu((E_Q^\omega)_-)} - \frac{1_{(E_Q^\omega)_+}}{\mu((E_Q^\omega)_+)}\right),$$

where

$$m(E_Q^\omega) = \frac{\mu((E_Q^\omega)_-)\,\mu((E_Q^\omega)_+)}{\mu(E_Q^\omega)} = \left(\frac{1}{\mu((E_Q^\omega)_-)} + \frac{1}{\mu((E_Q^\omega)_+)}\right)^{-1}$$
$$\approx \min\{\mu((E_Q^\omega)_-), \mu((E_Q^\omega)_+)\}.$$

Note that this makes sense provided $\mu((E_Q^\omega)_-)\,\mu((E_Q^\omega)_+) > 0$. For otherwise, we set $h_Q^\omega \equiv 0$.

Note that for a fixed $Q \in \mathscr{D}$ and $\omega \in \mathscr{W}(Q)$, one can easily verify that h_Q^ω satisfies the properties (a)–(d) in Definition 2.7. Let us further observe that h_Q^ω is orthogonal to $h_Q^{\omega'}$ for $\omega \neq \omega'$. We would like to emphasize that here we have $2^d - 1$ generalized Haar functions associated to each Q (one for each $\omega \in \mathscr{W}(Q)$). In this way, if for every Q we pick $\omega_Q \in \mathscr{W}(Q)$, we have that $\{h_Q^{\omega_Q}\}_{Q \in \mathscr{D}}$ is a 2-value generalized Haar system in $\mathbb{R}^d$ (see Definition 2.7 and Remark 2.8) and therefore standard (see (2.12)).

Mitrea's Haar system is constructed in the following way. Let us fix an enumeration $(Q_j)_{j=1}^{2^d}$ of the dyadic children of Q. For every $2 \leq j \leq 2^d$ we set $\widetilde{Q}_j = \cup_{k=j}^{2^d} Q_k$. We define Mitrea's Haar system as follows: for every $1 \leq j \leq 2^d - 1$ we set

$$H_Q^j = \sqrt{m(Q_j)}\left(\frac{1_{Q_j}}{\mu(Q_j)} - \frac{1_{\widetilde{Q}_{j+1}}}{\mu(\widetilde{Q}_{j+1})}\right),$$

where

$$m(Q_j) = \frac{\mu(Q_j)\,\mu(\widetilde{Q}_{j+1})}{\mu(\widetilde{Q}_j)} = \left(\frac{1}{\mu(Q_j)} + \frac{1}{\mu(\widetilde{Q}_{j+1})}\right)^{-1} \approx \min\{\mu(Q_j), \mu(\widetilde{Q}_{j+1})\}.$$

This definition makes sense provided $\mu(Q_j)\,\mu(\widetilde{Q}_{j+1}) > 0$. For otherwise, we set $H_Q^j \equiv 0$.

Again, for a fixed $1 \leq j \leq 2^d - 1$ and $Q \in \mathscr{D}$, one can easily verify that H_Q^j satisfies the properties (a)–(d) in Definition 2.7 and also that H_Q^j is orthogonal to $H_Q^{j'}$ for $j \neq j'$. As before, we have $2^d - 1$ generalized Haar functions associated to each Q (one for each j). Hence, if for every Q we pick j_Q, $1 \leq j_Q \leq 2^d - 1$, we have that $\{H_Q^{j_Q}\}_{Q\in\mathscr{D}}$ is a 2-value generalized Haar system in $\mathbb{R}^d$ (see Definition 2.7 and Remark 2.8) and therefore standard (see (2.12)).

We finally present another way to construct Haar systems in the spirit of the wavelet construction. For this example, we assume that μ is a product measure, that is, $\mu = \mu_1 \times \cdots \times \mu_d$ where $\mu_1, \ldots, \mu_d$ are Borel measures in $\mathbb{R}$ satisfying $\mu_j(I) < \infty$ for every $I \in \mathscr{D}$. We will use the following notation, given $Q \in \mathscr{D}(\mathbb{R}^d)$ we have that $Q = I_1^Q \times \cdots \times I_d^Q$ with $I_j^Q \in \mathscr{D}(\mathbb{R})$. Hence, $\mu(Q) = \prod_{j=1}^d \mu_j(I_j^Q)$. Associated to each μ_j we consider a μ_j-generalized Haar system $\Phi_j = \{\phi_{j,I}^1\}_{I\in\mathscr{D}(\mathbb{R})}$. For every $I \in \mathscr{D}(\mathbb{R})$ with $\mu_j(I) > 0$ we set $\phi_{j,I}^0 = 1_I/\mu_j(I)^{\frac{1}{2}}$ and $\phi_{j,I}^0 \equiv 0$ otherwise. For every $\epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \{0,1\}^d \setminus \{0\}^d$ and $Q \in \mathscr{D}(\mathbb{R}^d)$ we define

$$\phi_Q^\epsilon(x) = \prod_{j=1}^d \phi_{j,I_j^Q}^{\epsilon_j}(x_j).$$

We have that each ϕ_Q^ϵ satisfies the properties (a)–(d) in Definition 2.7 and also that ϕ_Q^ϵ is orthogonal to $\phi_Q^{\epsilon'}$ for $\epsilon \neq \epsilon'$. Hence, if for every Q we pick ϵ_Q, as above, we have that $\{\phi_Q^{\epsilon_Q}\}_{Q\in\mathscr{D}}$ is a generalized Haar system in $\mathbb{R}^d$, see Definition 2.7. Note that Remark 2.8 says each Φ_j is a 2-value generalized Haar system in $\mathbb{R}$. However, unless some further condition is imposed in each measure μ_j, one has that ϕ_Q^ϵ may take more than 2 non-vanishing values (this is quite easy if we take $\epsilon = \{1\}^d$). Nevertheless, if $Q \in \mathscr{D}_\Phi$ then

$$\|\phi_Q^\epsilon\|_{L^1(\mu)} = \prod_{j=1}^d \|\phi_{j,I_j^Q}^{\epsilon_j}\|_{L^1(\mu_j)}, \qquad \|\phi_Q^\epsilon\|_{L^\infty(\mu)} = \prod_{j=1}^d \|\phi_{j,I_j^Q}^{\epsilon_j}\|_{L^\infty(\mu_j)}.$$

Let $m_j(I) = \mu_j(I_-)\mu_j(I_+)/\mu_j(I)$ for $I \in \mathscr{D}_{\Phi_j}$. Then we have that, for every $I \in \mathscr{D}_{\Phi_j}$,

$$\|\phi_{j,I}^0\|_{L^1(\mu_j)} = \sqrt{\mu_j(I)}, \qquad \|\phi_{j,I}^0\|_{L^\infty(\mu_j)} = \frac{1}{\sqrt{\mu_j(I)}},$$

and, as in 2.3,

$$\|\phi_{j,I}^1\|_{L^1(\mu_j)} = 2\sqrt{m_j(I)}, \qquad \|\phi_{j,I}^1\|_{L^\infty(\mu_j)} \approx \frac{1}{\sqrt{m_j(I)}}.$$

Thus, despite the fact that Φ is not a 2-value generalized Haar system in general, we obtain that Φ is standard.

To conclude this section we observe that although the generalized Haar systems we have constructed above are all standard, this is not the case in general. We work in $\mathbb{R}^2$ and for $k \geq 2$ we let $Q_k = [k, k+1) \times [k, k+1)$. Fix an enumeration $Q_k^1, Q_k^2, Q_k^3, Q_k^4$ of the dyadic children of Q_k. Define $F(x) \equiv 1$ if $x \notin \cup_{k \geq 2} Q_k$ and elsewhere

$$F(x) = \sum_{k=2}^{\infty} \Big(\frac{4}{k^2} \big(1_{Q_k^1}(x) + 1_{Q_k^2}(x) \big) + \frac{2(k^2 - 2)}{k^2} \big(1_{Q_k^3}(x) + 1_{Q_k^4}(x) \big) \Big).$$

We consider $d\mu(x) = F(x)\,dx$ which is a Borel measure such that $0 < \mu(Q) < \infty$ for every $Q \in \mathscr{D}$. By construction we have

$$\mu(Q_k^1) = \mu(Q_k^2) = \frac{1}{k^2}, \qquad \mu(Q_k^3) = \mu(Q_k^4) = \frac{k^2 - 2}{2k^2}, \qquad \mu(Q_k) = 1.$$

Next we consider the system $\Phi = \{\phi_{Q_k}\}_{k \geq 2}$ with

$$\phi_{Q_k} = \frac{1}{2k} \Big(\frac{1_{Q_k^1}}{\mu(Q_k^1)} - \frac{1_{Q_k^2}}{\mu(Q_k^2)} \Big) + \sqrt{\frac{k^2 - 2}{8\,k^2}} \Big(\frac{1_{Q_k^3}}{\mu(Q_k^3)} - \frac{1_{Q_k^4}}{\mu(Q_k^4)} \Big)$$

$$= \frac{k}{2} \big(1_{Q_k^1} - 1_{Q_k^2} \big) + \sqrt{\frac{k^2}{2\,(k^2 - 2)}} \big(1_{Q_k^3} - 1_{Q_k^4} \big).$$

By construction each ϕ_{Q_k} satisfies (a)–(d) in Definition 2.7 where we observe that in (d) we have $\|\phi_{Q_k}\|_{L^2(\mu)} = 1$. Thus, Φ is a generalized Haar system in $\mathbb{R}^2$. On the other hand,

$$\|\phi_{Q_k}\|_{L^1(\mu)} \|\phi_{Q_k}\|_{L^\infty(\mu)} = \Big(\frac{1}{k} + \sqrt{\frac{k^2 - 2}{2\,k^2}} \Big) \max \Big\{ \frac{k}{2}, \sqrt{\frac{k^2}{2\,(k^2 - 2)}} \Big\}$$

$$\geq \sqrt{\frac{k^2 - 2}{2\,k^2}} \frac{k}{2} = \frac{\sqrt{k^2 - 2}}{2\sqrt{2}} \xrightarrow[k \to \infty]{} \infty.$$

Therefore, Φ is not standard. We note that in view of Example 2.14 we have that the Haar multiplier

$$(4.3) \qquad\qquad T_\epsilon f = \sum_{Q \in \mathscr{D}} \epsilon_Q \langle f, \phi_Q \rangle \phi_Q, \qquad \epsilon_Q = \pm 1$$

is not of weak-type $(1, 1)$. We can obtain this from Theorem 2.11. However, here the situation is very simple: we just take $\varphi_{Q_k} = 1_{Q_k^1} / \mu(Q_k^1)$ and obtain that

$$T_\epsilon \varphi_{Q_k} = \epsilon_{Q_k} \langle \varphi_{Q_k}, \phi_{Q_k} \rangle \phi_{Q_k} = \epsilon_{Q_k} \frac{k}{2} \phi_{Q_k}.$$

Thus,

$$\frac{\|T_\epsilon \varphi_{Q_k}\|_{L^{1,\infty}(\mu)}}{\|\varphi_{Q_k}\|_{L^1(\mu)}} \approx \frac{k \, \|\phi_{Q_k}\|_{L^1(\mu)}}{2} = \frac{k}{2} \Big(\frac{1}{k} + \sqrt{\frac{k^2 - 2}{2\,k^2}} \Big) \xrightarrow[k \to \infty]{} \infty,$$

and therefore T_ϵ is not of weak-type $(1, 1)$.

Let us finally point out that in the classical situation (i.e., when μ is the Lebesgue measure and we take a standard Haar system) these operators

are usually referred to as a martingale transforms. As it is well known, martingale transforms are of weak-type $(1,1)$ for any measure μ by the use of probability methods. Surprisingly, T_ϵ is not of weak-type $(1,1)$ and therefore T_ϵ cannot be written as a "martingale transform" operator in terms of martingale differences (see (5.15) below for further details).

4.3. **Examples of measures in higher dimensions.**

Taking into account the previous constructions, we are going to give some examples of non trivial measures so that the conditions in Theorem 2.11 hold. We first notice that if μ is dyadically doubling then $\mu(Q) \approx \mu(Q')$ for every dyadic children Q' of Q. In particular, for any generalized Haar system Φ, one can show that $\|\Phi_Q\|_{L^1(\mu)} \approx \mu(Q)^{1/2}$ and $\|\Phi_Q\|_{L^\infty(\mu)} \approx \mu(Q)^{-1/2}$ for every $Q \in \mathscr{D}_\Phi$. This clearly implies that we always have that $\Xi(\Phi, \Psi; r, s) \le C_{r,s}$ for any choices of generalized Haar systems. Thus, the problem becomes interesting when μ is not dyadically doubling. The general case admits too many choices, and we just want to give an illustration of the kind of issues that one can find. Therefore we are going to restrict ourselves to dimension $d = 2$ with $0 < \mu(Q) < \infty$ for every $Q \in \mathscr{D}(\mathbb{R}^2)$ and $\Phi = \Psi$ with $\mathscr{D}_\Phi = \mathscr{D}$. We are going to consider the complexities $(1,0)$ and $(0,1)$ (since these are related to the model operators $H_\mathscr{D}$ and $H_\mathscr{D}^*$ in 1-dimension).

We consider Wilson's construction. We halve each Q horizontally and write Q_N for the northern "hemisphere" and Q_S the southern "hemisphere". If for every cube Q we take the anti-clockwise enumeration starting with the west-south corner then $Q_S = E_Q^{\{1,2\}}$ and $Q_N = E_Q^{\{3,4\}}$. We now take Wilson's system $\Phi = \{h_Q^{\{1,2,3,4\}}\}_{Q \in \mathscr{D}}$, that is,

$$h_Q^{\{1,2,3,4\}} = \sqrt{m_{N,S}(Q)} \left(\frac{1_{Q_S}}{\mu(Q_S)} - \frac{1_{Q_N}}{\mu(Q_N)}\right), \qquad m_{N,S}(Q) = \frac{\mu(Q_S)\,\mu(Q_N)}{\mu(Q)}.$$

Suppose that $d\mu(x,y) = dx\,d\nu(y)$ then μ is dyadically doubling iff ν is dyadically doubling. If $Q = I \times J$ then

$$m_{N,S}(Q) = |I|\,m_\nu(J) = |I|\,\frac{\nu(J_-)\,\nu(J_+)}{\nu(J)}.$$

Then $\Xi(\Phi, \Phi; 0, 1) < \infty$ if and only if ν is m_ν-increasing and $\Xi(\Phi, \Phi; 1, 0) < \infty$ if and only if ν is m_ν-decreasing. Using the examples we constructed above we find measures μ in $\mathbb{R}^2$ which are non-dyadically doubling but they satisfy one (or both) conditions.

However if we use another Haar system we get a different behavior. Suppose now that our enumeration is clockwise and starts with the west-south corner then $Q_W = E_Q^{\{1,2\}}$ and $Q_E = E_Q^{\{3,4\}}$ are respectively the western and eastern "hemispheres". If now take Wilson's system $\Phi = \{h_Q^{\{1,2,3,4\}}\}$ then we get the same definitions as before replacing Q_S by Q_W and Q_N by Q_E. In particular,

$$m_{E,W}(Q) = \frac{|I|}{4}\,\nu(J)$$

Then we always have $\Xi(\Phi, \Phi; 0, 1) \le 1/\sqrt{2} < \infty$, whereas $\Xi(\Phi, \Phi; 1, 0) < \infty$ if and only if ν is dyadically doubling.

Similar examples can be constructed using Mitrea's Haar shifts.

We finally look at the Haar system using the wavelet construction. If our system is comprised of $\phi_Q^{i,1}(x,y) = \phi_{1,I}^i(x)\phi_{2,J}^1(y)$ with $i = 0$ or 1 we obtain

$$\|\phi_Q^{i,1}\|_{L^1(dx \times d\nu)} = 2\sqrt{|I|\, m_\nu(J)}, \qquad \|\phi_Q^{i,1}\|_{L^\infty(dx \times d\nu)} \approx \frac{1}{\sqrt{|I|\, m_\nu(J)}},$$

and then we have the same behavior as before: $\Xi(\Phi, \Phi; 0, 1) < \infty$ if and only if ν is m_ν-increasing and $\Xi(\Phi, \Phi; 1, 0) < \infty$ if and only if ν is m_ν-decreasing. On the other hand, if we take $\phi_Q^{1,0}(x,y) = \phi_{1,I}^1(x)\phi_{2,J}^0(y)$ and obtain

$$\|\phi_Q^{1,0}\|_{L^1(dx \times d\nu)} = \sqrt{|I|\, \nu(J)}, \qquad \|\phi_Q^{1,0}\|_{L^\infty(dx \times d\nu)} = \frac{1}{\sqrt{|I|\, \nu(J)}}.$$

Then we always have $\Xi(\Phi, \Phi; 0, 1) \leq 1/\sqrt{2} < \infty$, whereas $\Xi(\Phi, \Phi; 1, 0) < \infty$ if and only if ν is dyadically doubling.

5. Further Results

5.1. **Non-cancellative Haar shift operators.** One can consider Haar shift operators defined in terms of generalized Haar systems that are not required to satisfy the vanishing integral condition. To elaborate on this, let us first consider the case of the dyadic paraproducts and their adjoints. The space $\mathrm{BMO}_{\mathscr{D}}(\mu)$ is the space of locally integrable functions ρ such that

$$\|\rho\|_{\mathrm{BMO}_{\mathscr{D}}(\mu)} = \sup_{Q \in \mathscr{D}} \Big(\frac{1}{\mu(Q)} \int_Q |\rho(x) - \langle \rho \rangle_Q|^2 \, d\mu(x) \Big)^{\frac{1}{2}} < \infty,$$

where as usual the terms where $\mu(Q) = 0$ are assumed to be 0. Given $\rho \in \mathrm{BMO}_{\mathscr{D}}(\mu)$, and $\Theta = \{\theta_Q\}_{Q \in \mathscr{D}}$, $\Psi = \{\psi_Q\}_{Q \in \mathscr{D}}$, two (cancellative) generalized Haar systems, we define the *dyadic paraproduct* Π_ρ:

$$\Pi_\rho f(x) = \sum_{Q \in \mathscr{D}} \langle \rho, \theta_Q \rangle \langle f \rangle_Q \psi_Q(x).$$

Note that for each cube Q, θ_Q and ψ_Q are cancellative generalized Haar functions. However, the term $\langle f \rangle_Q$ can be viewed, after renormalization, as f paired with the non-cancellative generalized Haar function $1_Q/\mu(Q)^{1/2}$. That is the reason why we call this operator a non-cancellative Haar shift, see below for further details.

Alternatively, one can consider dyadic paraproducts by incorporating μ-Carleson sequences. Given a sequence $\gamma = \{\gamma_Q\}_{Q \in \mathscr{D}}$, we say that γ is a μ-Carleson sequence, which is denoted by $\gamma \in \mathscr{C}(\mu)$, if for every $Q \in \mathscr{D}$ we have that $\gamma_Q = 0$ if $\mu(Q) = 0$ and

$$\|\gamma\|_{\mathscr{C}(\mu)} = \sup_{Q \in \mathscr{D}, \, \mu(Q) > 0} \Big(\frac{\sum_{Q' \in \mathscr{D}(Q)} |\gamma_{Q'}|^2}{\mu(Q)} \Big)^{\frac{1}{2}} < \infty.$$

Typical examples of μ-Carleson sequences are given by $\mathrm{BMO}(\mu)$ functions. Indeed if $\rho \in \mathrm{BMO}_{\mathscr{D}}(\mu)$, $\Theta = \{\theta_Q\}_{Q \in \mathscr{D}}$ is a generalized Haar system and we set $\gamma_Q = \langle \rho, \theta_Q \rangle$ we have that γ is μ-Carleson measure: if $Q_0 \in \mathscr{D}$ such that $\mu(Q_0) > 0$, we have by orthogonality

$$\sum_{Q\in\mathscr{D}(Q_0)} |\gamma_Q|^2 = \sum_{Q\in\mathscr{D}(Q_0)} |\langle(\rho - \langle\rho\rangle_{Q_0})1_{Q_0}, \theta_Q\rangle|^2$$

$$\leq \|(\rho - \langle\rho\rangle_{Q_0})1_{Q_0}\|_{L^2(\mu)}^2 \leq \|\rho\|_{\mathrm{BMO}(\mu)}^2 \mu(Q_0)$$

and therefore $\|\gamma\|_{\mathscr{C}(\mu)} \leq \|\rho\|_{\mathrm{BMO}(\mu)}$. One can also reverse this procedure. Indeed, given $\gamma \in \mathscr{C}(\mu)$ and a generalized Haar system $\Theta = \{\theta_Q\}_{Q\in\mathscr{D}}$ we can define a function ρ which is a Haar expansion using Θ with the coefficients given by the sequence γ as follows. It suffices to consider the function ρ in any d-dimensional quadrant, say for simplicity that we are in $\mathbb{R}_1^d = [0,\infty)^d$. Let $Q_k = [0, 2^{-k})^d$ and set

$$\rho(x) = \sum_{k\in\mathbb{Z}} \Big(\sum_{Q\in\mathscr{D}(Q_k)\setminus\mathscr{D}(Q_{k+1})} \gamma_Q\, \theta_Q(x)\Big)1_{Q_k\setminus Q_{k+1}}(x).$$

Note that for every $x \in \mathbb{R}_1^d$, the sum in k contains only one non-vanishing term. From orthogonality and the Carleson condition it follows that for every $k_0 \in \mathbb{Z}$,

$$\|\rho\|_{L^2(Q_{k_0})}^2 \leq \sum_{k\geq k_0} \sum_{Q\in\mathscr{D}(Q_k)\setminus\mathscr{D}(Q_{k+1})} |\gamma_Q|^2 = \sum_{Q\in\mathscr{D}(Q_{k_0})} |\gamma_Q|^2 \leq \|\gamma\|_{\mathscr{C}(\mu)}^2 \mu(Q_{k_0}).$$

In particular ρ is locally integrable. We next take an arbitrary $R \in \mathscr{D}$, $R \subset \mathbb{R}_1^d$. Then there exists a unique k such that $R \subset Q_k \setminus Q_{k+1}$. Then for every $x \in R$ we have

$$\rho(x) = \sum_{Q\in\mathscr{D}(Q_k)\setminus\mathscr{D}(Q_{k+1})} \gamma_Q\, \theta_Q(x)$$

$$= \sum_{Q\in\mathscr{D}(R)} \gamma_Q\, \theta_Q(x) + \sum_{\substack{Q\in\mathscr{D}(Q_k)\setminus\mathscr{D}(Q_{k+1})\\ R\subsetneq Q}} \gamma_Q\, \theta_Q(x) = I(x) + II.$$

Note that II is constant and that $\int_R I(x)\, d\mu(x) = 0$ then

$$\int_R |\rho(x) - \langle\rho\rangle_R|^2\, d\mu(x) = \int_R \Big| \sum_{Q\in\mathscr{D}(R)} \gamma_Q\, \theta_Q(x)\Big|^2 d\mu(x)$$

$$\leq \sum_{Q\in\mathscr{D}(R)} |\gamma_Q|^2 \leq \|\gamma\|_{\mathscr{C}(\mu)}^2 \mu(R).$$

This implies that $\rho \in \mathrm{BMO}(\mu)$ with $\|\rho\|_{\mathrm{BMO}(\mu)} \leq \|\gamma\|_{\mathscr{C}(\mu)}$. Further details are left to the reader.

Given γ a μ-Carleson sequence and $\Psi = \{\psi_Q\}_{Q\in\mathscr{D}}$ a generalized Haar system we define the *dyadic paraproduct* Π_γ as follows

$$\Pi_\gamma f(x) = \sum_{Q\in\mathscr{D}} \gamma_Q \langle f\rangle_Q \psi_Q(x).$$

If we set $\widetilde{\phi}_Q = 1_Q/\mu(Q)^{1/2}$ if $\mu(Q) > 0$ and $\widetilde{\phi}_Q \equiv 0$ otherwise we have that $\widetilde{\Phi} = \{\widetilde{\phi}_Q\}_{Q\in\mathscr{D}}$ satisfies (a), (b) and (d) in Definition 2.7. Since (c) does not hold we call $\widetilde{\Phi}$ a non-cancellative generalized Haar system. In such a way we can write

$$\Pi_\gamma f(x) = \sum_{Q\in\mathscr{D}} \alpha_Q \langle f, \widetilde{\phi}_Q\rangle \psi_Q(x), \qquad \alpha_Q = \frac{\gamma_Q}{\mu(Q)^{\frac{1}{2}}}.$$

Note that

$$|\alpha_Q| \le \left(\frac{\sum_{Q' \in \mathscr{D}(Q)} |\gamma_{Q'}|^2}{\mu(Q)} \right)^{\frac{1}{2}} \le \|\gamma\|_{\mathscr{C}(\mu)}.$$

Thus, we can see Π_γ as a Haar shift of complexity $(0,0)$ with respect to the non-cancellative generalized Haar system $\widetilde{\Phi}$ and the (cancellative) generalized Haar system Ψ. Notice that the adjoint of the paraproduct can be written as

$$\Pi_\gamma^* f(x) = \sum_{Q \in \mathscr{D}} \gamma_Q \langle f, \psi_Q \rangle \frac{1_Q(x)}{\mu(Q)} = \sum_{Q \in \mathscr{D}} \alpha_Q \langle f, \psi_Q \rangle \widetilde{\phi}_Q(x).$$

Again Π_γ^* is a Haar shift of complexity $(0,0)$ with respect to a (cancellative) generalized Haar system Ψ and the non-cancellative generalized Haar system $\widetilde{\Phi}$. This motivates the definition of a non-cancellative Haar shift operator:

$$(5.1) \quad \widetilde{\amalg}_{r,s} f(x) = \sum_{Q \in \mathscr{D}} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \alpha_{R,S}^Q \langle f, \widetilde{\phi}_R \rangle \widetilde{\psi}_S(x), \qquad \sup_{Q,R,S} |\alpha_{R,S}^Q| < \infty,$$

with $\widetilde{\Phi} = \{\widetilde{\phi}_Q\}_{Q \in \mathscr{D}}$ and $\widetilde{\Psi} = \{\widetilde{\psi}_Q\}_{Q \in \mathscr{D}}$ being two non-cancellative generalized Haar systems, i.e., both of them satisfies (a), (b) and (d) in Definition 2.7. We would like to stress that $\widetilde{\Phi}$ and $\widetilde{\Psi}$ do not necessarily satisfy (c), therefore the $L^2(\mu)$ boundedness does not automatically follow from the assumed conditions. Thus, is natural to impose that $\widetilde{\amalg}_{r,s}$ is bounded on $L^2(\mu)$ along with some local boundedness property and these condition will be checked in any specific situation.

Theorem 5.2. *Let μ be a Borel measure on $\mathbb{R}^d$, $d \ge 1$, satisfying that $\mu(Q) < \infty$ for every $Q \in \mathscr{D}$. Let $\widetilde{\Phi} = \{\widetilde{\phi}_Q\}_{Q \in \mathscr{D}}$ and $\widetilde{\Psi} = \{\widetilde{\psi}_Q\}_{Q \in \mathscr{D}}$ be two non-cancellative generalized Haar systems in $\mathbb{R}^d$. Let r, s be two non-negative integers and consider $\widetilde{\amalg}_{r,s}$ as in (5.1). Assume that $\widetilde{\amalg}_{r,s}$ is bounded on $L^2(\mu)$ and also that $\widetilde{\amalg}_{r,s}$ satisfies the following restricted local $L^2(\mu)$ boundedness: for every $Q_0 \in \mathscr{D}$ we have that*

$$(5.3) \qquad \|\widetilde{\amalg}_{r,s}^{Q_0}(1_{Q_0})\|_{L^2(\mu)} \lesssim \mu(Q_0)^{\frac{1}{2}},$$

where the constant is uniform on Q_0 and

$$\widetilde{\amalg}_{r,s}^{Q_0} f(x) = \sum_{Q \in \mathscr{D}(Q_0)} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \alpha_{R,S}^Q \langle f, \widetilde{\phi}_R \rangle \widetilde{\psi}_S(x).$$

If $\Xi(\widetilde{\Phi}, \widetilde{\Psi}; r, s) < \infty$, then $\widetilde{\amalg}_{r,s}$ maps continuously $L^1(\mu)$ into $L^{1,\infty}(\mu)$.

Remark 5.4. Let us observe that $\widetilde{\amalg}_{r,s}^{Q_0}$ is the non-cancellative Haar shift operator associated with the sequence $\gamma_{R,S}^Q = \alpha_{R,S}^Q$ for $Q \in \mathscr{D}(Q_0)$, $R \in \mathscr{D}_r(Q)$, $S \in \mathscr{D}_s(Q)$; and $\gamma_{R,S}^Q = 0$ otherwise. Also, the $L^2(\mu)$ boundedness of $\widetilde{\amalg}_{r,s}^{Q_0}$ clearly implies (5.3).

Remark 5.5. Notice that if we further assume that both Haar systems $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are cancellative, then we automatically obtain (5.3) and the $L^2(\mu)$ boundedness of $\widetilde{\amalg}_{r,s}^{Q_0}$ (see Section 3.3). In such a case Theorem 5.2 becomes Theorem 2.11

Proof. The proof is similar to that of Theorem 2.11, therefore we only give the parts of the argument that are different. Again we may assume that $\mu \in \mathcal{M}_\infty$, the general case follows as before. Follow the proof of Theorem 2.11. For S_1 we use our assumption that $\widetilde{\text{Ш}}_{r,s}$ is bounded on $L^2(\mu)$. The estimate for S_2 is the same. Let us observe that the estimate for S_3 is entirely analogous since in (3.7) we have not used the vanishing integral of ϕ_Q. We are then left with estimating S_4, for which we first observe that

$$S_4 \leq \mu(\Omega_\lambda) + \mu\{x \in \mathbb{R}^d \setminus \Omega_\lambda : |\widetilde{\text{Ш}}_{r,s}\beta(x)| > \lambda/3\}$$
$$\leq \frac{1}{\lambda}\|f\|_{L^1(\mu)} + \frac{3}{\lambda}\sum_j \Big(\int_{\mathbb{R}^d \setminus \widehat{Q}_j} |\widetilde{\text{Ш}}_{r,s}\beta_j| d\mu + \int_{\widehat{Q}_j \setminus Q_j} |\widetilde{\text{Ш}}_{r,s}\beta_j| d\mu \Big)$$

and we estimate each term in the interior sum. Proceeding as in (3.7) and using Theorem 2.1 we can analogously obtain

$$\sum_j \int_{\mathbb{R}^d \setminus \widehat{Q}_j} |\widetilde{\text{Ш}}_{r,s}\beta_j| \, d\mu$$
$$\lesssim \sum_j \sum_{\widehat{Q}_j \subsetneq Q \subset Q_j^{(r+1)}} \sum_{\substack{R \in \mathscr{D}_r(Q), R \subset \widehat{Q}_j \\ S \in \mathscr{D}_s(Q)}} \|\beta_j\|_{L^1(\mu)} \|\widetilde{\phi}_R\|_{L^\infty(\mu)} \|\widetilde{\psi}_S\|_{L^1(\mu)}$$
$$\leq 2^{2+(r+s)d} r \, \Xi(\widetilde{\Phi}, \widetilde{\Psi}; r, s)\|f\|_{L^1(\mu)}.$$

On the other hand, for every $x \in \widehat{Q}_j \setminus Q_j$ we have

$$|\widetilde{\text{Ш}}_{r,s}\beta_j(x)| \leq \Big| \sum_{Q \subsetneq \widehat{Q}_j} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \cdots \Big| + \Big| \sum_{\widehat{Q}_j \subset Q} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \cdots \Big| = F_j(x) + G_j(x)$$

and we estimate each function in turn. For $F_j(x)$ we note that the terms $Q \subset Q_j$ vanish and therefore $R \subset Q \subset \widehat{Q}_j \setminus Q_j$. Thus β_j is constant on R and then

$$F_j(x) = \Big| \sum_{Q \subsetneq \widehat{Q}_j \setminus Q_j} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \alpha_{R,S}^Q \langle \beta_j, \widetilde{\phi}_R \rangle \, \widetilde{\psi}_S(x) \Big|$$
$$= |\langle f \rangle_{Q_j} - \langle f \rangle_{\widehat{Q}_j}| \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)} \Big| \sum_{Q \subsetneq \widehat{Q}_j \setminus Q_j} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \alpha_{R,S}^Q \langle 1_{\widehat{Q}_j \setminus Q_j}, \widetilde{\phi}_R \rangle \, \widetilde{\psi}_S(x) \Big|$$
$$\leq 2 \langle |f| \rangle_{Q_j} \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)} \sum_{\substack{Q' \in \mathscr{D}_1(\widehat{Q}_j) \\ Q' \neq Q_j}} \Big| \sum_{Q \in \mathscr{D}(Q')} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \alpha_{R,S}^Q \langle 1_{Q'}, \widetilde{\phi}_R \rangle \, \widetilde{\psi}_S(x) \Big|$$
$$= 2 \langle |f| \rangle_{Q_j} \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)} \sum_{\substack{Q' \in \mathscr{D}_1(\widehat{Q}_j) \\ Q' \neq Q_j}} |\widetilde{\text{Ш}}_{r,s}^{Q'}(1_{Q'})(x)|.$$

This, the fact that $\text{supp}\,\widetilde{\text{Ш}}_{r,s}^{Q'}(1_{Q'}) \subset Q'$ and that these cubes and pairwise disjoint, and (5.3) yield

$$\int_{\widehat{Q}_j \setminus Q_j} F_j \, d\mu \leq 2 \langle |f| \rangle_{Q_j} \left(\frac{\mu(Q_j)}{\mu(\widehat{Q}_j)} \right) \sum_{\substack{Q' \in \mathscr{D}_1(\widehat{Q}_j) \\ Q' \neq Q_j}} \int_{Q'} |\widetilde{\mathrm{III}}_{r,s}^{Q'}(1_{Q'})| d\mu$$

$$\leq 2 \langle |f| \rangle_{Q_j} \left(\frac{\mu(Q_j)}{\mu(\widehat{Q}_j)} \right) \sum_{\substack{Q' \in \mathscr{D}_1(\widehat{Q}_j) \\ Q' \neq Q_j}} \|\widetilde{\mathrm{III}}_{r,s}^{Q'}(1_{Q'})\|_{L^2(\mu)} \, \mu(Q')^{\frac{1}{2}} \lesssim \int_{Q_j} |f| \, d\mu.$$

For G_j we proceed as before

$$\int_{\widehat{Q}_j \setminus Q_j} G_j \, d\mu \lesssim \sum_{\widehat{Q}_j \subset Q \subset Q_j^{(r+1)}} \sum_{\substack{R \in \mathscr{D}_r(Q), R \subset \widehat{Q}_j \\ S \in \mathscr{D}_s(Q)}} \|\beta_j\|_{L^1(\mu)} \|\tilde{\phi}_R\|_{L^\infty(\mu)} \|\tilde{\psi}_S\|_{L^1(\mu)}$$

$$\leq 2^{(r+s)\,d}\, r\, \Xi(\widetilde{\Phi}, \widetilde{\Psi}; r, s) \|\beta_j\|_{L^1(\mu)}.$$

Gathering the previous estimates we conclude that

$$\sum_j \int_{\widehat{Q}_j \setminus Q_j} |\widetilde{\mathrm{III}}_{r,s} \beta_j| \, d\mu \leq \sum_j \int_{\widehat{Q}_j \setminus Q_j} (F_j + G_j) \, d\mu$$

$$\lesssim \sum_j \int_{Q_j} |f| \, d\mu + \sum_j \|\beta_j\|_{L^1(\mu)} \lesssim \|f\|_{L^1(\mu)}.$$

$\square$

Remark 5.6. As above, if we keep track of the constants and use a standard homogeneity argument we obtain that, under the conditions of Theorem 2.11,

$$\|\widetilde{\mathrm{III}}_{r,s}\|_{L^1(\mu) \to L^{1,\infty}(\mu)} \leq C_0 \left(\|\widetilde{\mathrm{III}}_{r,s}\|_{L^2(\mu) \to L^2(\mu)} \right.$$

$$\left. + \sup_{Q \in \mathscr{D}, \mu(Q) \neq 0} \frac{\|\widetilde{\mathrm{III}}_{r,s}^Q(1_Q)\|_{L^2(\mu)}}{\sqrt{\mu(Q)}} + 2^{(s+r)\,d}\, r\, \Xi(\Phi, \Psi; r, s) \sup_{Q,R,S} |\alpha_{R,S}^Q| \right),$$

where C_0 is a universal constant (independent of the dimension, for instance, in the previous argument one can safely take $C_0 \leq 220$).

5.2. Dyadic paraproducts.

As a consequence of Theorem 5.2 we can obtain the following result for dyadic paraproducts.

Theorem 5.7. *Let μ be a Borel measure on $\mathbb{R}^d$, $d \geq 1$, satisfying that $\mu(Q) < \infty$ for every $Q \in \mathscr{D}$. Let $\Psi = \{\psi_Q\}_{Q \in \mathscr{D}}$ be a generalized Haar system. Given a sequence $\gamma = \{\gamma_Q\}_{Q \in \mathscr{D}}$ we consider the* dyadic paraproduct Π_γ *and its adjoint Π_γ^*:*

$$\Pi_\gamma f(x) = \sum_{Q \in \mathscr{D}} \gamma_Q \langle f \rangle_Q \psi_Q(x), \qquad \Pi_\gamma^* f(x) = \sum_{Q \in \mathscr{D}} \gamma_Q \langle f, \psi_Q \rangle \frac{1_Q(x)}{\mu(Q)}.$$

Then we have the following:

(i) *For every $\gamma \in \mathscr{C}(\mu)$, Π_γ is of weak-type $(1,1)$ and there exists a universal constante C_0 (one can take for instance $C_0 \leq 288$) such that*

$$\|\Pi_\gamma f\|_{L^{1,\infty}(\mu)} \leq C_0 \|\gamma\|_{\mathscr{C}(\mu)} \|f\|_{L^1(\mu)}.$$

Consequently, Π_γ is bounded on $L^p(\mu)$, $1 < p \le 2$ (the constant is dimension free and depends linearly on $\|\gamma\|_{\mathscr{C}(\mu)}$).

(ii) *If*

$$\text{(5.8)} \qquad \sup_{Q \in \mathscr{D}} \|\psi_Q\|_{L^\infty(\mu)} \, \mu(Q)^{\frac{1}{2}} < \infty,$$

then Π_γ^* is of weak-type $(1,1)$ for every $\gamma \in \mathscr{C}(\mu)$ with boundedness constant depending linearly on $\|\gamma\|_{\mathscr{C}(\mu)}$. Conversely, if Π_γ^* is of weak-type $(1,1)$ with $\|\Pi_\gamma^*\|_{L^1(\mu) \to L^{1,\infty}} \le C \|\gamma\|_{\mathscr{C}(\mu)}$ for every $\gamma \in \mathscr{C}(\mu)$, then (5.8) holds. Additionally, if (5.8) holds then Π_γ^* is bounded on $L^p(\mu)$ for $1 < p < 2$ (the case $p \ge 2$ follows from (i) without assuming (5.8)).

(iii) *Suppose in particular that $d = 1$, $\mu(I) > 0$ for every $I \in \mathscr{D}$ and that $\Psi = \mathcal{H}$. Then, Π_γ is of weak-type $(1,1)$ and bounded on $L^p(\mu)$, $1 < p \le 2$, for every $\gamma \in \mathscr{C}(\mu)$. However, if for every $\gamma \in \mathscr{C}(\mu)$ we have that Π_γ^* is of weak-type $(1,1)$ or weak-type (p,p) for some $1 < p < 2$, then μ is dyadically doubling. Conversely, if μ is dyadically doubling then Π_γ^* is of weak-type $(1,1)$ and bounded on $L^p(\mu)$, $1 < p < 2$, for every $\gamma \in \mathscr{C}(\mu)$.*

(iv) *In (i), (ii), (iii) we can replace the condition "$\gamma \in \mathscr{C}(\mu)$" by "$\gamma_Q = \langle \rho, \theta_Q \rangle$ with $\rho \in \mathrm{BMO}(\mu)$ and $\Theta = \{\theta_Q\}_{Q \in \mathscr{D}}$ a generalized Haar system"; and in the boundedness constants $\|\gamma\|_{\mathscr{C}(\mu)}$ by $\|\rho\|_{\mathrm{BMO}(\mu)}$.*

Before starting the proof, let us state the $L^2(\mu)$ boundedness of the paraproduct (and its adjoint) along with the corresponding restricted local boundedness as a lemma:

Lemma 5.9. *Under the assumptions of Theorem 5.7, for every $\gamma \in \mathscr{C}(\mu)$ we have*

$$\text{(5.10)} \qquad \|\Pi_\gamma f\|_{L^2(\mu)} \le 2 \, \|\gamma\|_{\mathscr{C}(\mu)} \, \|f\|_{L^2(\mu)}.$$

Moreover, for every $Q_0 \in \mathscr{D}$ we obtain

$$\text{(5.11)} \qquad \|\Pi_\gamma^{Q_0} f\|_{L^2(\mu)} \le 2 \, \|\gamma\|_{\mathscr{C}(\mu)} \, \|f\|_{L^2(\mu)}, \quad \Pi_\gamma^{Q_0} f = \sum_{Q \in \mathscr{D}(Q_0)} \gamma_Q \, \langle f \rangle_Q \psi_Q.$$

Proof. We claim that it suffices to obtain (5.10). Indeed, we consider a new sequence $\widetilde{\gamma} = \{\widetilde{\gamma}_Q\}_{Q \in \mathscr{D}}$ with $\widetilde{\gamma}_Q = \gamma_Q$ if $Q \in \mathscr{D}(Q_0)$ and $\widetilde{\gamma}_Q = 0$ otherwise. We clearly have that $\widetilde{\gamma} \in \mathscr{C}(\mu)$ with $\|\widetilde{\gamma}\|_{\mathscr{C}(\mu)} \le \|\gamma\|_{\mathscr{C}(\mu)}$ and also $\Pi_\gamma^{Q_0} = \Pi_{\widetilde{\gamma}}$. Thus, (5.10) applied to $\widetilde{\gamma}$ implies (5.11).

We obtain (5.10) using ideas from [17]. Let us first suppose that $\mu \in \mathcal{M}_\infty$. The argument is somehow standard, but, since our setting is very general, we give the argument for completeness. Given $f \in L^2(\mu)$ and $\lambda > 0$, as in Theorem 2.1, we can find a maximal collection of dyadic cubes $\{Q_j^\lambda\}_j$ such that $\Omega_\lambda = \cup_j Q_j^\lambda$. We notice that the existence of such maximal cubes follows from the fact that $\langle |f| \rangle_Q \le \langle |f|^2 \rangle_Q^{1/2} \to 0$ as $\ell(Q) \to \infty$, given our current assumption $\mu \in \mathcal{M}_\infty$. Next we use that Ψ is cancellative, therefore orthogonal,

$$(5.12) \quad \|\Pi_\gamma f\|_{L^2(\mu)}^2 = \sum_{Q\in\mathscr{D}} |\gamma_Q|^2 \, |\langle f\rangle_Q|^2 \|\psi_Q\|_{L^2(\mu)}^2$$

$$\leq \int_0^\infty \sum_{Q\in\mathscr{D}} 1_{\{\langle |f|\rangle_Q > \lambda\}} |\gamma_Q|^2 \, 2\lambda \, d\lambda \leq \int_0^\infty \sum_j \sum_{Q\in\mathscr{D}(Q_j^\lambda)} |\gamma_Q|^2 \, 2\lambda \, d\lambda.$$

$$\leq \|\gamma\|_{\mathscr{C}(\mu)}^2 \int_0^\infty \sum_j \mu(Q_j^\lambda) \, 2\lambda \, d\lambda \leq \|\gamma\|_{\mathscr{C}(\mu)}^2 \int_0^\infty \mu(\Omega_\lambda) \, 2\lambda \, d\lambda$$

$$= \|\gamma\|_{\mathscr{C}(\mu)}^2 \, \|M_\mathscr{D} f\|_{L^2(\mu)}^2 \leq 4 \, \|\gamma\|_{\mathscr{C}(\mu)}^2 \, \|f\|_{L^2(\mu)}^2,$$

and this completes the proof of the fact that Π_γ is bounded on $L^2(\mu)$ provided $\mu \in \mathcal{M}_\infty$. To consider the general case, as before we may suppose that $\operatorname{supp} f \subset \mathbb{R}_k^d$, $1 \leq k \leq 2^d$ with $\mu(\mathbb{R}_k^d) < \infty$. In (5.12) we split the integral in two: $0 < \lambda \leq \langle |f|\rangle_{\mathbb{R}_k^d}$ and $\lambda > \langle |f|\rangle_{\mathbb{R}_k^d}$. In the second case we can find the maximal cubes $\{Q_j^\lambda\}$ and the previous argument goes through. Let us next consider the integral in the range $0 < \lambda \leq \langle |f|\rangle_{\mathbb{R}_k^d}$. Let $\{Q_n\}_{n\geq 1} \subset \mathscr{D}(\mathbb{R}_k^d)$ be an increasing sequence such that $\cup_n Q_n = \mathbb{R}_k^d$. Then, we proceed as above

$$\int_0^{\langle |f|\rangle_{\mathbb{R}_k^d}} \sum_{Q\in\mathscr{D}(\mathbb{R}_k^d)} 1_{\{\langle |f|\rangle_Q > \lambda\}} |\gamma_Q|^2 \, 2\lambda \, d\lambda \leq \langle |f|\rangle_{\mathbb{R}_k^d}^2 \sup_n \sum_{Q\in\mathscr{D}(Q_n)} |\gamma_Q|^2$$

$$\leq \|\gamma\|_{\mathscr{C}(\mu)}^2 \, \langle |f|^2\rangle_{\mathbb{R}_k^d} \sup_n \mu(Q_n) = \|\gamma\|_{\mathscr{C}(\mu)}^2 \, \|f\|_{L^2(\mu)}^2.$$

This completes the proof of (5.10). □

Proof of Theorem 5.7. We start with Π_γ. Set $\widetilde{\phi}_Q = 1_Q/\mu(Q)^{1/2}$ if $\mu(Q) > 0$ and $\widetilde{\phi}_Q = 0$ otherwise and consider the non-cancellative generalized Haar system $\widetilde{\Phi} = \{\widetilde{\phi}_Q\}_{Q\in\mathscr{D}}$. As explained above, in the notation of Theorem 5.2, Π_γ is a non-cancellative Haar shift operator of complexity $(0,0)$ with respect to the systems $\widetilde{\Phi}$ and $\widetilde{\Psi} = \Psi$. By Lemma 5.9 we have the required $L^2(\mu)$ bounds in Theorem 5.2. Thus the weak-type $(1,1)$ (and by interpolation the boundedness on $L^p(\mu)$, $1 < p < 2$) of Π_γ follows from the property $\Xi(\widetilde{\Phi}, \Psi; 0, 0) < \infty$. But this is in turn trivial: by Hölder's inequality we have for every $Q \in \mathscr{D}_\Psi$

$$\|\widetilde{\phi}_Q\|_{L^\infty(\mu)}\|\psi_Q\|_{L^1(\mu)} = \mu(Q)^{-\frac{1}{2}}\|\psi_Q\|_{L^1(\mu)} \leq \|\psi_Q\|_{L^2(\mu)} = 1.$$

This completes the proof of (i). For the boundedness constant we can use Remark 5.6 along with Lemma 5.9 to obtain the linear dependence on $\|\gamma\|_{\mathscr{C}(\mu)}$.

We now turn to (ii). We have shown that Π_γ is bounded on $L^2(\mu)$ and so is its adjoint Π_γ^*. Notice that $(\Pi_\gamma^*)^{Q_0} = (\Pi_\gamma^{Q_0})^*$ and therefore $(\Pi_\gamma^*)^{Q_0}$ satisfies (5.11). Then, we apply again Theorem 5.2 to Π_γ^* which is a non-cancellative Haar shift operator of complexity $(0,0)$ with respect to the non-cancellative generalized Haar systems Ψ, $\widetilde{\Phi}$. Thus, $\Xi(\Psi, \widetilde{\Phi}; 0, 0) < \infty$, which coincides with (5.8), implies that Π_γ^* is of weak-type $(1,1)$. The linear dependence on $\|\gamma\|_{\mathscr{C}(\mu)}$ uses the same argument as above. Let us now obtain the converse. Notice that in (5.8) we can restrict the supremum to $Q \in \mathscr{D}_\Psi$ and in particular $\mu(Q) > 0$. Fix one of these cubes Q_0 and let

$\gamma_Q = \delta_{Q,Q_0} \sqrt{\mu(Q_0)}$. Then, $\gamma \in \mathscr{C}(\mu)$ with $\|\gamma\|_{\mathscr{C}(\mu)} = 1$. Take

$$f = \mathrm{sgn}\left(\psi_{Q_0}(x)\right) \frac{1_{Q_{0,\infty}}(x)}{\mu(Q_{0,\infty})},$$

where $Q_{0,\infty} \in \mathscr{D}_1(Q_0)$ is a cube where ψ_{Q_0} attains its maximum. Then, as in the proof of Theorem 2.11 and using that Π_γ^* is of weak-type $(1,1)$ with uniform constant (since $\|\gamma\|_{\mathscr{C}(\mu)} = 1$) we obtain

$$\|\psi_{Q_0}\|_{L^\infty(\mu)} \sqrt{\mu(Q_0)} = \left\| \langle f, \psi_{Q_0} \rangle \frac{1_{Q_0}}{\sqrt{\mu(Q_0)}} \right\|_{L^{1,\infty}(\mu)} = \|\Pi_\gamma^* f\|_{L^{1,\infty}(\mu)}$$
$$\leq C \|f\|_{L^1(\mu)} = C.$$

Repeating this for every $Q_0 \in \mathscr{D}_\Psi$ we obtain (5.8) as desired.

To complete the proof of (ii) we first observe that for $p \geq 2$, duality and (i) give the $L^p(\mu)$ boundedness of Π_γ^* with no further assumption on μ. For $1 < p < 2$, assuming (5.8), we already know that Π_γ^* is of weak-type $(1,1)$. The desired estimates now follow by interpolation with the $L^2(\mu)$ bound from Lemma 5.9.

To obtain (iii) we apply (i) and (ii) and observe that (5.8) can be written as

$$\sup_{I \in \mathscr{D}} \frac{\sqrt{\mu(I)}}{\sqrt{\min\{\mu(I_-), \mu(I_+)\}}} \approx \sup_{I \in \mathscr{D}} \frac{\sqrt{\mu(I)}}{\sqrt{m(I)}} < \infty,$$

which in turn is equivalent to the fact that μ is dyadically doubling. To complete the proof of (iii) it remains to show that if Π_γ^* is of weak type (p,p) for some $1 < p < 2$ then μ is dyadically doubling. Fix then $1 < p < 2$ and $I_0 \in \mathscr{D}$. Let $\gamma_I = \delta_{I,\widehat{I_0}} \sqrt{\mu(\widehat{I_0})}$ and observe that $\gamma \in \mathscr{C}(\mu)$ with $\|\gamma\|_{\mathscr{C}(\mu)} = 1$. Taking $f = h_{\widehat{I_0}}$, by (3.5) we have

$$\mu(\widehat{I_0})^{\frac{1}{p} - \frac{1}{2}} = \left\| \gamma_{\widehat{I_0}} \frac{1_{\widehat{I_0}}}{\mu(\widehat{I_0})} \right\|_{L^{p,\infty}(\mu)} = \|\Pi_\gamma^* f\|_{L^{p,\infty}(\mu)}$$
$$\leq C \|f\|_{L^p(\mu)} \approx m(\widehat{I_0})^{\frac{1}{2} - \frac{1}{p'}} = m(\widehat{I_0})^{\frac{1}{p} - \frac{1}{2}} \leq \mu(I_0)^{\frac{1}{p} - \frac{1}{2}}.$$

This estimate holds for every $I_0 \in \mathscr{D}$ and therefore μ is dyadically doubling as desired.

We finally show (iv). As observed before if we set $\gamma_Q = \langle \rho, \theta_Q \rangle$ with $\rho \in \mathrm{BMO}(\mu)$ and $\Theta = \{\theta_Q\}_{Q \in \mathscr{D}}$ being a generalized Haar system we have that $\gamma \in \mathscr{C}(\mu)$ with $\|\gamma\|_{\mathscr{C}(\mu)} \leq \|\rho\|_{\mathrm{BMO}(\mu)}$. Therefore the only assertion that is not contained in the previous items is the converse implication in (ii). As before, in (5.8), we can restrict the supremum to $Q \in \mathscr{D}_\Psi$ and in particular $\mu(Q) > 0$. Fix one of these cubes Q_0, take $\Theta = \Psi$ and let $\rho = \psi_{Q_0} \sqrt{\mu(Q_0)}$. Then,

$$\|\rho\|_{\mathrm{BMO}_{\mathscr{D}}(\mu)}^2 = \sup_{Q_0 \subset Q \in \mathscr{D}} \frac{1}{\mu(Q)} \int_Q \left| \rho - \langle \rho \rangle_Q \right|^2 d\mu = \sup_{Q_0 \subset Q \in \mathscr{D}} \frac{\mu(Q_0)}{\mu(Q)} = 1.$$

We take the same function f as in (ii), use that Π_ρ^* is of weak-type $(1,1)$ with uniform constant (since $\|\rho\|_{\mathrm{BMO}_{\mathscr{D}}(\mu)} = 1$) and obtain

$$\|\psi_{Q_0}\|_{L^\infty(\mu)}\, \mu(Q_0)^{\frac{1}{2}} = \Big\|\langle f, \psi_{Q_0}\rangle\, \frac{1_{Q_0}}{\sqrt{\mu(Q_0)}}\Big\|_{L^{1,\infty}(\mu)} = \|\Pi_\rho^* f\|_{L^{1,\infty}(\mu)}$$

$$\le C\, \|f\|_{L^1(\mu)} = C.$$

Repeating this for every $Q_0 \in \mathscr{D}_\Psi$ we obtain as desired (5.8). This completes the proof of (iv). $\qquad\square$

5.3. **On the probabilistic approach.** We shall work with a fixed Borel measure μ on $\mathbb{R}^d$ such that $\mu(Q) < \infty$ for every dyadic cube Q. The dyadic system $\mathscr{D} = (\mathscr{D}_k)_{k\in\mathbb{Z}}$ is a filtration on $\mathbb{R}^d$. The conditional expectation operator E_k associated to $\mathscr{D}_k$ is defined by

$$\mathrm{E}_k f(x) = \sum_{Q\in\mathscr{D}_k} \mathrm{E}_Q f(x) = \sum_{Q\in\mathscr{D}_k} \langle f\rangle_Q 1_Q(x),$$

where $\langle f\rangle_Q = 0$ if $\mu(Q) = 0$. The martingale difference operators D_k are given by $\mathrm{D}_k = \mathrm{E}_k - \mathrm{E}_{k-1}$. It is clear from the definitions that the operators E_k form an increasing family projections that preserve integrals and that D_k are orthogonal projections. Thus, if $f \in L^p(\mu)$, $1 \le p < \infty$, the sequence $(\mathrm{E}_k f)_{k\in\mathbb{Z}}$ is an L^p-martingale and

$$(5.13) \qquad f(x) = \sum_{k\in\mathbb{Z}} \mathrm{D}_k f + \mathrm{E}_{-\infty} f = \sum_{n>k} \mathrm{D}_n f + \mathrm{E}_k f,$$

where the convergence is in $L^p(\mu)$ and μ-almost everywhere, and where $\mathrm{E}_{-\infty} f = \sum_{j=1}^{2^d} \langle f\rangle_{\mathbb{R}_j^d} 1_{\mathbb{R}_j^d}$. Let $Q \in \mathscr{D}_{k-1}$ and denote by D_Q the projection

$$D_Q f(x) = \mathrm{D}_k f(x)\, 1_Q(x) = \Big(\sum_{Q'\in\mathscr{D}_1(Q)} \mathrm{E}_{Q'} f(x)\Big) - \mathrm{E}_Q f(x).$$

Hence $\mathrm{D}_k = \sum_{Q\in\mathscr{D}_{k-1}} \mathrm{D}_Q$. Observe that we may set $\mathrm{D}_Q f \equiv 0$ if $\mu(Q) = 0$. We easily obtain that $\phi \in \mathrm{D}_Q(L^2(\mu))$ (by this we mean the image of $L^2(\mu)$ by the operator D_Q) if and only if ϕ is supported on Q, constant on dyadic subcubes of Q, and has vanishing μ-integral. In such a case we may write

$$(5.14) \qquad \phi(x) = \sum_{Q'\in\mathscr{D}_1(Q)} a_{Q'} \frac{1_{Q'}(x)}{\mu(Q')},$$

with $\sum_{Q'\in\mathscr{D}_1(Q)} a_{Q'} = 0$, and where it is understood that $a_Q' = 0$ if $\mu(Q') = 0$ and we use the standard convention that $0 \cdot \infty = 0$. Hence, $\mathrm{D}_Q(L^2(\mu))$ is a vector space of dimension at most $2^d - 1$.

If we are in dimension $d = 1$ and $I \in \mathscr{D}$ satisfies $\mu(I) > 0$, then $h_I \in \mathrm{D}_I(L^2(\mu))$ (since $\mathrm{D}_I h_I = h_I$). Note that in such a case $\mathrm{D}_I(L^2(\mu))$ is 1-dimensional and therefore $\mathrm{D}_I f = \langle f, h_I\rangle h_I$, for every $f \in L^2(\mu)$.

In the higher dimensional case, assume for simplicity that $\mu(Q) > 0$ for every $Q \in \mathscr{D}$. Let us consider the Wilson's Haar system $\{h_Q^\omega : \omega \in \mathscr{W}(Q), Q \in \mathscr{D}\}$. By othonormality of the Wilson's Haar system and the fact that the cardinality of $\mathscr{W}(Q)$ is $2^d - 1$ we immediately obtain that $\{h_Q^\omega : \omega \in \mathscr{W}(Q)\}$ is an orthonormal basis of $\mathrm{D}_Q(L^2(\mu))$. Thus,

$$\mathrm{D}_Q f = \sum_{\omega\in\mathscr{W}(Q)} \langle f, h_Q^\omega\rangle h_Q^\omega, \qquad f \in L^2(\mu).$$

The same can be done with Mitrea's Haar system (see above), in which case we obtain

$$\mathrm{D}_Q f = \sum_{j=1}^{2^d-1} \langle f, H_Q^j \rangle H_Q^j, \qquad f \in L^2(\mu).$$

Finally, if $\mu = \mu_1 \times \cdots \times \mu_d$ with μ_j Borel measures in $\mathbb{R}$ such that $0 < \mu_j(I) < \infty$ for every $I \in \mathscr{D}(\mathbb{R})$ and we consider the Haar system in the spirit of the wavelet construction $\{\phi_Q^\epsilon : \epsilon \in \{0,1\}^d \setminus \{0\}^d, Q \in \mathscr{D}\}$ we analogously have

$$\mathrm{D}_Q f = \sum_{\epsilon \in \{0,1\}^d \setminus \{0\}^d} \langle f, \phi_Q^\epsilon \rangle \phi_Q^\epsilon, \qquad f \in L^2(\mu).$$

We next see that martingale transforms can be written as Haar multipliers (i.e., Haar shifts of complexity $(0,0)$). A martingale transform is defined as

$$T f(x) = \sum_{k \in \mathbb{Z}} \xi_k(x) \, \mathrm{D}_k f(x)$$

where the sequence $\{\xi_k\}_{k \in \mathbb{Z}}$ is predictable with respect to the dyadic filtration $(\mathscr{D}_k)_{k \in \mathbb{Z}}$, that is, ξ_k is $\sigma(\mathscr{D}_{k-1})$-measurable. Then ξ_k is constant on the cubes $Q \in \mathscr{D}_{k-1}$. Namely, $\xi_k(x) = \sum_{Q \in \mathscr{D}_{k-1}} \alpha_Q 1_Q(x)$. Thus, by definition of the projections D_Q we get then that the martingale transform defined by $\{\xi_k\}_{k \in \mathbb{Z}}$ can be equivalently written as

$$(5.15) \quad T f(x) = \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathscr{D}_{k-1}} \alpha_Q \, \mathrm{D}_Q f(x) = \sum_{Q \in \mathscr{D}} \alpha_Q \left(\sum_{j=1}^{2^d-1} \langle f, \psi_Q^j \rangle \psi_Q^j(x) \right),$$

with $\{\psi_Q^j\}_{1 \le j \le 2^d-1}$ being any orthonormal basis of $\mathrm{D}_Q(L^2(\mu))$. Thus, every martingale transform can be represented as a sum of $2^d - 1$ Haar multipliers, i.e., a Haar shift operators of complexity $(0,0)$ (see Example 2.14). Note that each Haar shift operator in the sum is written in terms of the system $\{\psi_Q^{j_Q}\}_{Q \in \mathscr{D}}$ where for each $Q \in \mathscr{D}$ we chose j_Q with $1 \le j_Q \le 2^d - 1$.

It is easy to see that any orthonormal basis $\{\psi_Q^j\}_{1 \le j \le 2^d-1}$ of $\mathrm{D}_Q(L^2(\mu))$ is also a basis of $\mathrm{D}_Q(L^p(\mu))$ for $1 \le p \le \infty$. Assuming further that $\mu \in \mathcal{M}_\infty$, (5.13) says that $\{\psi_Q^j\}_{1 \le j \le 2^d-1, Q \in \mathscr{D}}$ is a basis of $L^p(\mu)$, $1 \le p < \infty$. However, in view of (5.15), Burkholder's theorem of L^p boundedness of martingale transforms, $1 < p < \infty$, does not suffice to show that a given Haar basis is unconditional in $L^p(\mu)$. In fact, unconditionality of a Haar basis is not true in general. We take the last example in Section 4.2 of a non-standard generalized Haar system and the Haar multiplier in (4.3). We can easily see that for every $1 < p < 2$,

$$\frac{\|T_\epsilon \varphi_{Q_k}\|_{L^p(\mu)}}{\|\varphi_{Q_k}\|_{L^p(\mu)}} = \frac{k \, \|\phi_{Q_k}\|_{L^p(\mu)}}{2 \, \mu(Q_k^1)^{\frac{1}{p}-1}} = \frac{k}{2 \, k^2 \frac{p-1}{p}} \left(\frac{k^{p-2}}{2^{p-1}} + \left(\frac{k^2}{k^2-2} \right)^{\frac{p}{2}-1} 2^{-\frac{p}{2}} \right)^{\frac{1}{p}}$$

$$\gtrsim k^{\frac{2-p}{p}} \xrightarrow[k \to \infty]{} \infty.$$

Also, if we now take $\widetilde{\varphi}_{Q_k} = 1_{Q_k^3}/\mu(1_{Q_k^3})$ then, for $2 < p < \infty$,

$$\frac{\|T_\epsilon \widetilde{\varphi}_{Q_k}\|_{L^p(\mu)}}{\|\widetilde{\varphi}_{Q_k}\|_{L^p(\mu)}} = \left(\frac{k^2}{2\,(k^2-2)}\right)^{\frac{1}{2}} \frac{\|\phi_{Q_k}\|_{L^p(\mu)}}{\mu(Q_k^3)^{\frac{1}{p}-1}}$$

$$= 2^{\frac{1}{p}-\frac{3}{2}} \left(\frac{k^2-2}{k^2}\right)^{\frac{1}{2}-\frac{1}{p}} \left(\frac{k^{p-2}}{2^{p-1}} + \left(\frac{k^2}{k^2-2}\right)^{\frac{p}{2}-1} 2^{-\frac{p}{2}}\right)^{\frac{1}{p}} \gtrsim k^{\frac{p-2}{p}} \xrightarrow[k\to\infty]{} \infty.$$

These imply that $\Phi = \{\phi_{Q_k}\}_{k\geq 2}$ is not an unconditional basis (on its span) on $L^p(\mu)$ for $1 < p < \infty$ with $p \neq 2$.

Nevertheless, the standardness property

$$\sup_{1\leq j\leq 2^d-1} \sup_{Q\in\mathscr{D}} \|\psi_Q^j\|_{L^1(\mu)} \|\psi_Q^j\|_{L^\infty(\mu)} < \infty,$$

implies, by Theorem 2.11, that every Haar multiplier is of weak type $(1,1)$ and, by interpolation and duality, $L^p(\mu)$ bounded for every $1 < p < \infty$. This, in turn, gives that $\{\psi_Q^j\}_{1\leq j\leq 2^d-1, Q\in\mathscr{D}}$ is an unconditional basis for $L^p(\mu)$, $1 < p < \infty$.

Let us now look at the case of the dyadic Hilbert transform an its adjoint in dimension $d = 1$. Assume that $\mu(I) > 0$ for every $I \in \mathscr{D}$. One can easily see that

$$h_{I_\pm}(x) = \mp \frac{\mu(I_\pm)}{\sqrt{m(I)}} h_{I_\pm}(x) h_I(x).$$

Hence,

$$H_{\mathscr{D}} f(x) = \sum_{I\in\mathscr{D}} \langle f, h_I\rangle (h_{I_-}(x) - h_{I_+}(x))$$

$$= \sum_{I\in\mathscr{D}} \frac{1}{\sqrt{m(I)}} (\mu(I_-)h_{I_-}(x) + \mu(I_+)h_{I_+}(x))\langle f, h_I\rangle h_I(x)$$

$$= \sum_{k\in\mathbb{Z}} \left(\sum_{I\in\mathscr{D}_{k-1}} \frac{\mu(I_-)h_{I_-}(x) + \mu(I_+)h_{I_+}(x)}{\sqrt{m(I)}}\right) \left(\sum_{J\in\mathscr{D}_{k-1}} \langle f, h_J\rangle h_J(x)\right)$$

$$= \sum_{k\in\mathbb{Z}} \xi_k(x) \mathrm{D}_k f(x),$$

where we have used that $\mathrm{D}_k = \sum_{I\in\mathscr{D}_{k-1}} \mathrm{D}_I$ and that $\mathrm{D}_I f = \langle f, h_I\rangle h_I$. The coefficient ξ_k is $k+1$-measurable, defining a non predictable sequence. One may thus regard the dyadic Hilbert transform as a "generalized martingale transform". Let us finally observe that for the adjoint of the Hilbert transform, since D_k is a projection, we have

$$H_{\mathscr{D}}^* f(x) = \sum_{k\in\mathbb{Z}} \mathrm{D}_k(\xi_k f)(x).$$

Similar expressions can be obtained for other Haar shift operators in every dimension provided the coefficients can be split as $\alpha_{R,S}^Q = \gamma_R^Q \beta_S^Q$. This procedure shows that Haar shift operators of arbitrary complexity "fill" the space of "martingale transforms" with arbitrary measurable coefficients, further details are left to the interested reader. In particular, we see why classical tools coming from martingale L^p-theory do not apply in the present contexts, and our Calderón-Zygmund decomposition establishes the right substitute of Gundy's martingale decomposition in such a general setting.

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